

# Optimal Control for Geometric Motion Planning of a Robot Diver

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**Abstract**—Inertial reorientation of articulated bodies has been an active area of research in the robotics community, as this behavior can help guide dynamic robots to a safe landing with minimal damage. The main objective of this work is emulating the aggressive and large angle correction maneuvers, like somersaults, that are performed by human divers. To this end, a planar three link robot, called DiverBot, is proposed. By considering a gravity-free scenario, a *local connection* is obtained between joint angles and the body orientation, resulting in a reduction in the system dynamics. An optimal control policy applied on this reduced configuration space yielded diving maneuvers that appear human-like while being dynamically feasible. Numerical results show that the DiverBot can execute one somersault without drift and multiple somersaults with drift.

## I. INTRODUCTION

Mid-air reorientation is an exciting domain of research for roboticists, who are currently trying to achieve this behavior in robots of various morphologies [1], [2], [3], [4], [5]. Animals use this ability to self-right their orientations during free-falls to protect themselves from severe, and sometimes fatal, injuries. Once robots leave the protective and controlled research lab environments and go outdoors, it would be imperative for them to know how to fall in ways that cause minimal damage.

The main aim of this work is to achieve large mid-air reorientation maneuvers using a planar three-linked robot, called DiverBot, which is modeled after a human diver. Though diving is not necessarily a damage control action, the maneuver serves as an ideal benchmark to characterize the quick reorienting power of any control law developed for a human-like robot. While humans have a remarkable ability to naturally perform multiple flips during the diving manoeuvre, the underlying biological control is not fully understood, and therefore hard to emulate in robots.

The falling cat problem is probably the most popular and well-studied instance of mid-air reorientation from the animal kingdom, specifically of releasing with zero angular momentum (also called *drift-less*) and allowing the use of only internal body shape changes. Kane and Scher [6] proposed the first dynamical solution to this problem, while Montgomery [7] explained this manoeuvre in further detail using the concept of a *local connection*. This local connection relates the shape changes of cat’s limbs to the net rotation of its body, as a consequence of the conservation of net angular momentum [8], [9]. When seen from this standpoint, such systems are classical examples of non-holonomic

systems, whose motion planning has been explored in the controls community in the past [10], [11], [12].

Similarly, human diving action could also be explained by invoking the conservation of angular momentum [13]. Modeling and control of human diving was also studied within the robotics and control community. In [14], the diving is divided into phases and appropriate control laws are used in each phase. On the contrary, in both [15] and [16], the non-holonomic nature of the system is exploited to propose simple learning-based and bang-bang control laws, respectively. Finally, a successful implementation of a somersault, using a 3D biped was demonstrated in [17], using a phased controller and launching from the ground with non-zero momentum. While all these controllers provide valid solutions, they do not comment on the optimality of the maneuver. More recently however, sensorimotor control theories have emerged which conclude that our movement coordination does indeed seek to achieve an *optimal performance* [18], [19]. Some optimal control formulations have been proposed for diving in the past [20], [21], [22], albeit without explicitly exploiting the local connection.

The main aim for this work is to develop a geometric mechanics-based optimal control law that is able to generate human-like diving as a single end-to-end motion. This allows us to posit that the naturally occurring human diving maneuver is an optimal response to the rapid mid-air reorientation problem. Moreover, we leverage the reduced form of the system dynamics to generate a smaller optimization problem and thus improve speed of convergence to a feasible solution. A systematic method to derive the reduced dynamics and pose an optimal control problem is provided.

The rest of the paper is organized as follows. Section II introduces the robot model and the corresponding local connection is derived. In Section III, the optimal control problem is formulated for the diving motion. Section IV summarizes key numerical simulation results. Finally, Section V provides conclusions and directions for future work.

## II. MODEL DESCRIPTION

For this study, we limit our exploration of diving to the planar case, to first characterize the dominant behavior along the sagittal plane. Moreover, the hand and leg are modeled as single links without the elbow and knee joints, respectively. Note that, reorientations are harder (more expensive) with this configuration as greater torque needs to be applied at the hip and shoulder joints for reducing inertia, during the maneuver. In this case, there is also a possibility for the system center of mass (CoM) to drift further away from the body link CoM. However, an optimal solution to the three-

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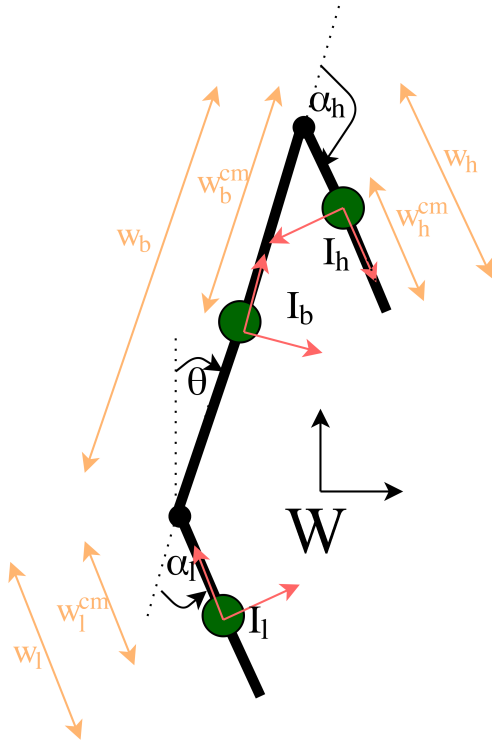


Fig. 1: A planar three link diver model

link model guarantees the existence of a feasible solution to its five link counterpart, and it also serves as an ideal initial seed to search for an optimal policy in that high dimensional space.

The robot model, as shown in Fig. 1, has a configuration manifold which is  $Q \in \text{SO}(2) \times (\mathbb{S}^1)^2 \times \mathbb{R}^2$ . Fixing the body-fixed frame on *link<sub>b</sub>* (body link), its orientation and positions along  $x$ - and  $y$ -axes are defined w.r.t the ground inertial frame  $W$  as  $\theta$ ,  $x$ ,  $y$ , respectively. The relative orientations of *link<sub>h</sub>* (hand) and *link<sub>l</sub>* (leg) are given by  $\alpha_h$  and  $\alpha_l$ , respectively. Only the relative joints  $\alpha_1$  and  $\alpha_3$  can be controlled, making the system an under-actuated one. All system parameter definitions and notations are summarized in Table I.

Using the generalized coordinates,  $\mathbf{q} = [x \ y \ \alpha_h \ \theta \ \alpha_l]^T$ , we define the Lagrangian,  $L(\mathbf{q}, \dot{\mathbf{q}}) = T - V$ , where  $T$  and  $V$  are the kinetic and potential energies of the system respectively.

$$T = \sum_i^{h,b,l} \frac{1}{2} m_i \dot{\mathbf{x}}_i^2 + \frac{1}{2} I_h \dot{\alpha}_h^2 + \frac{1}{2} I_b \dot{\theta}^2 + \frac{1}{2} I_l \dot{\alpha}_l^2 \quad (1)$$

$$V = \sum_i^{h,b,l} m_i g y_i \quad (2)$$

Here,  $g$  denotes acceleration due to gravity, and  $\mathbf{x}_i = [x_i \ y_i]^T$ , where  $x_i$  and  $y_i$  denote the CoM position of *link<sub>i</sub>*. Hereafter, we neglect the translational dynamics of the body CoM, i.e.  $\dot{\mathbf{x}}_b = [\dot{x} \ \dot{y}]^T = 0$ , to focus only on the dynamics of reorientation. This reduces the configuration space to  $Q_r \in \text{SO}(2) \times (\mathbb{S}^1)^2$ . Using the principle of least action,

the dynamics of the system are derived and represented in the compact form below:

$$\mathbf{D}(\mathbf{q}_r) \ddot{\mathbf{q}}_r + \mathbf{C}(\mathbf{q}_r, \dot{\mathbf{q}}_r) + \mathbf{G}(\mathbf{q}_r) = \mathbf{B} \mathbf{u} \quad (3)$$

where,  $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $\mathbf{q}_r = [\alpha_h \ \theta \ \alpha_l]^T$  and  $\mathbf{u} = [u_h \ u_l]^T$ .

We can see that, if  $\mathbf{G}(\mathbf{q}_r) = 0$ , i.e. if the potential energy is neglected, then it can be easily seen that the Lagrangian is independent of  $\theta$ . In other words, the Lagrangian has a symmetry group  $\mathbb{S}^1$ , implying an invariance to changes in  $\theta$ . Similar to [15], for the rest of this study, we assume that the effects of gravity on  $\theta$  are minimum and thus neglect potential energy. However, for a given drop height, the resulting time of flight can be used to determine the desired average body velocity,  $\dot{\theta}_{avg}^d$ . This can be strictly enforced in the optimal control problem.

As a result of the invariance, Noether's theorem predicts the conservation of a quantity, which is the diver's angular momentum in this case.

$$\frac{\partial L}{\partial \theta} = 0 \quad \text{and} \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = 0 \quad (4)$$

Therefore,  $\frac{\partial L}{\partial \dot{\theta}} = p$ . Substituting  $p$  in (3) followed by some computations gives,

$$p = \mathbb{I}_\theta(\alpha) \dot{\theta} + \mathbb{I}_\alpha(\alpha) \dot{\alpha} \quad (5)$$

$$\dot{\theta} = -\mathbb{I}_\theta(\alpha)^{-1} \mathbb{I}_\alpha(\alpha) \dot{\alpha} + \mathbb{I}_\theta(\alpha)^{-1} p \quad (6)$$

where  $\alpha = [\alpha_h \ \alpha_l]$ . Equation (6) is called the *reconstruction equation* in the geometric mechanics community [8]. Additionally,  $-\mathbb{I}_\theta(\alpha)^{-1} \mathbb{I}_\alpha(\alpha)$  and  $\mathbb{I}_\theta(\alpha)$  are the local forms of the *mechanical connection*, compactly denoted as  $\mathbf{A}_\theta(\alpha)$ , and *locked inertia tensor*, respectively. It can easily verified that both the local connection and the locked inertia tensor only depend on the shape variables,  $(\alpha, \dot{\alpha})$ . Substituting eqn. (6) in (3) gives the reduced set of equations of motion, as

TABLE I: Notations used in this section and their definitions

Notation	Definition
$W$	Inertial fixed frame
$L_h$	Body frame fixed at the CoM of <i>link<sub>h</sub></i>
$L_l$	Body frame fixed at the CoM of <i>link<sub>l</sub></i>
$B$	Body frame fixed at the CoM of <i>link<sub>b</sub></i>
$\theta \in \text{SO}(2)$	Orientation from $W$ to $B$
$\alpha_h \in \mathbb{S}^1$	Orientation from $B$ to $L_h$
$\alpha_l \in \mathbb{S}^1$	Orientation from $B$ to $L_l$
$u_i \in \mathbb{R}$	Torque applied at <i>link<sub>i</sub></i>
$i$	Index to denote hand ( $h$ ), body ( $b$ ) and leg ( $l$ ) links
$m_i \in \mathbb{R}$	Mass of <i>link<sub>i</sub></i>
$I_i \in \mathbb{R}^{3 \times 3}$	Inertia Matrix of <i>link<sub>i</sub></i> in body frame
$w_i \in \mathbb{R}$	Length of <i>link<sub>i</sub></i>
$w_i^{com} \in \mathbb{R}$	Distance to the CoM of <i>link<sub>i</sub></i> from its top end.

shown below.

$$\begin{aligned} \dot{\theta} &= \mathbf{A}_\theta(\boldsymbol{\alpha})\dot{\boldsymbol{\alpha}} + \mathbb{I}_\theta(\boldsymbol{\alpha})^{-1}p \\ \mathbf{D}_\alpha\ddot{\boldsymbol{\alpha}} &= -\mathbf{C}_\alpha(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) + \mathbf{B}_\alpha\mathbf{u} \end{aligned} \quad (7)$$

Here,  $\mathbf{D}_\alpha, \mathbf{C}_\alpha$  are the Inertia and Coriolis matrices corresponding only to the dynamics of the relative shape changes. Correspondingly, the reduced input matrix is  $\mathbf{B}_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . The components of  $\mathbb{I}_\theta(\boldsymbol{\alpha}), \mathbb{I}_\alpha(\boldsymbol{\alpha}), \mathbf{D}_\alpha, \mathbf{C}_\alpha$ , can be found in the Appendix. In the next section, we formulate an optimal control problem on these reduced dynamics with an objective to achieve flips with least control effort.

### III. OPTIMAL CONTROL FORMULATION

#### A. Optimization Problem Formulation

Without loss of generality, we can rewrite (7) as a set of first-order equations:

$$\dot{\theta} = \mathbf{A}_\theta(\boldsymbol{\alpha})\dot{\boldsymbol{\alpha}} + \mathbb{I}_\theta(\boldsymbol{\alpha})^{-1}p \quad (8)$$

$$\dot{\alpha}_h = u_h \quad (9)$$

$$\dot{\alpha}_l = u_l \quad (10)$$

Note that, in Equations (7) there are three states (two of them are independent) and two control inputs in the dynamics of the DiverBot. The objective of performing flips can be articulated in the optimal control language as finding the control inputs  $\mathbf{u}(t) = [u_h(t), u_l(t)]$ , in the timespan  $t \in [t_I, t_F]$ , that will drive the system from an initial configuration  $\mathbf{y}_I = \mathbf{y}(t_I) = [\theta_I, \alpha_{h,I}, \alpha_{l,I}, \dot{\alpha}_{h,I}, \dot{\alpha}_{l,I}]^T$  to a desired final configuration  $\mathbf{y}_F = \mathbf{y}(t_F) = [\theta_d, \alpha_{h,d}, \alpha_{l,d}, \dot{\alpha}_{h,I}, \dot{\alpha}_{l,I}]^T$ . Accordingly, we pose the optimal control problem to find an admissible time history of the control input  $\mathbf{u}(t), t \in [t_I, t_F]$  as:

$$\min_{\mathbf{y}, \mathbf{u}} \mathbf{J}(\cdot) = \Phi(\mathbf{y}_F) + \int_{t_0}^{t_F} \mathcal{L}(\mathbf{y}(t)) dt \quad (11)$$

subject to:

$$\mathbf{H}(\mathbf{y}(t_I), \mathbf{y}(t_F)) = 0 \quad (12)$$

$$\mathbf{G}(\mathbf{y}(t), \mathbf{u}(t)) \leq 0 \quad (13)$$

where  $\Phi(\cdot)$  is the terminal cost,  $\mathcal{L}(\cdot)$  is the running cost.  $\mathbf{H}(\cdot)$  are equality constraints that enforce system dynamics and boundary conditions.  $\mathbf{G}(\cdot)$  are nonlinear inequality constraints that define joint angle and torque limits. For the problem to be solvable we assume that  $\mathbf{J}(\cdot), \mathbf{H}(\cdot), \mathbf{G}(\cdot)$  are all smooth functions.

#### B. Discretization

The optimal control problem can be seen as a Nonlinear Programming (NLP) problem with an infinite number of variables [23]. To be numerically tractable, discretize the problem to obtain a finite-dimensional approximation, using  $N$  collocation points such that,

$$t_I = t_0 < t_1 < \dots < t_{N-1} = t_F. \quad (14)$$

For ease of notation we represent the value of state variables at each time step as  $\mathbf{y}^k \equiv \mathbf{y}(t_k)$  for  $k = 0, 1, \dots, N-1$ , and similarly for the others. We chose a trapezoidal integration

scheme with time-steps of length  $h$  to transcribe the dynamics from Equation (8) to (10) to obtain the following equality constraints:

$$(\theta^{k+1} - \theta^k) - \frac{h}{2}(\dot{\theta}^{k+1} + \dot{\theta}^k) = 0 \quad (15)$$

$$\dot{\theta}^{k+1} - (A(\boldsymbol{\alpha}^{k+1})\dot{\boldsymbol{\alpha}}^{k+1} + I^{-1}(\boldsymbol{\alpha}^{k+1})p) = 0, \quad (16)$$

$$(\boldsymbol{\alpha}^{k+1} - \boldsymbol{\alpha}^k) - \frac{h}{2}(\mathbf{u}^{k+1} + \mathbf{u}^k) = 0. \quad (17)$$

Additionally, the corresponding inequality constraints that represent joint and input limits are defined as:

$$\alpha_i^{min} \leq \alpha_i^k \leq \alpha_i^{max}, \quad (18)$$

$$\dot{\alpha}_i^{min} \leq \dot{\alpha}_i^k \leq \dot{\alpha}_i^{max}, \quad (19)$$

$$u_i^{min} \leq u_i^k \leq u_i^{max}. \quad (20)$$

where  $\forall k \in \{0, \dots, N-1\}$  and  $i \in \{h, l\}$ .

#### C. Implementation

We leveraged the *interior-point* algorithm from MATLAB's Optimization Toolbox [24], to solve the proposed optimal control problem on the discretized and reduced dynamics. We run two passes of our optimization with increasing grid size,  $N_{\text{grid}} \in \{15, 50\}$  and limit to a maximum of 3000 iterations per pass. Like any non-linear constrained optimization problem, the minimum obtained is only locally optimal. More interestingly, this optimizer doesn't necessarily require the initial seed to be dynamically feasible. The algorithm converges fairly quickly even with a random guess. In the next section, the various numerical experiments conducted and their results are summarized.

## IV. RESULTS AND DISCUSSION

#### A. Experiment setup

We strictly constrain the final time  $t_F$  to be equal to the time of flight  $t_{\text{fall}}$  for any given initial height  $h_0$  as shown in (21). Additionally, joint and actuator limits in summarized in Table II while the model parameters are provided in Table III. Note that, the model parameters are determined from the rigid-body model of a human diver, presented in [14], by lumping it into only three segments(hand,torso,leg).

$$t_F \leq t_{\text{fall}} \Rightarrow t_F \leq \sqrt{\frac{2h_0}{g}} \quad (21)$$

We consider the planar diver reorientation problem under two scenarios. One is where the system starts at rest with no angular velocity (i.e. drift-free). In this case the system is said to be a *purely mechanical system* [8]. In the second case, we consider a system with some initial angular velocity  $\dot{\theta}_0$  (i.e. with drift). As a final test, we apply the optimal joint angle

TABLE II: State and Input Limits

	$\alpha_1$	$\alpha_2$	$u_1, u_2$
min	0	0	-1500
max	$\pi$	$\frac{2}{3}\pi$	1500

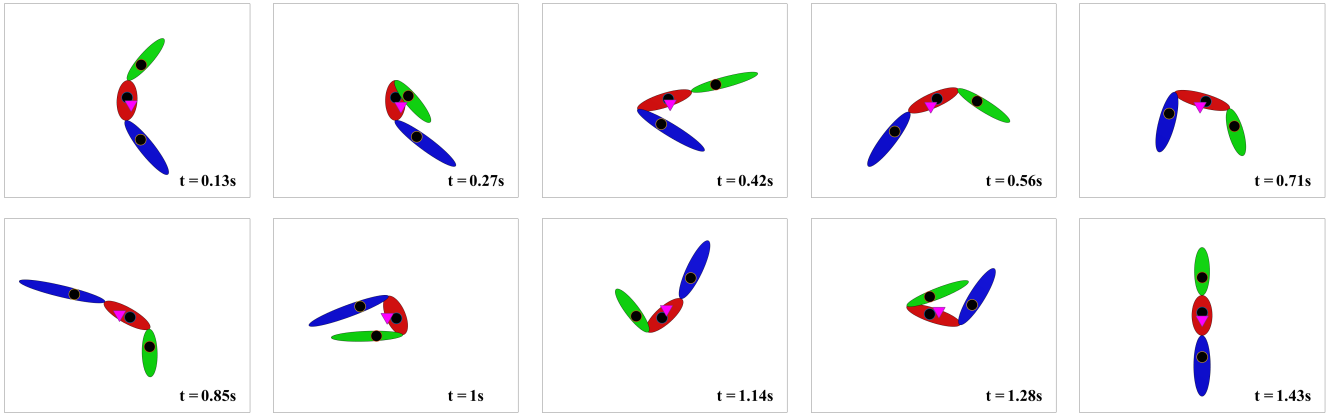


Fig. 2: Snapshots(with a time stamp) of DiverBot reorienting to perform a somersault simulation when dropped from a height of 10m without drift .

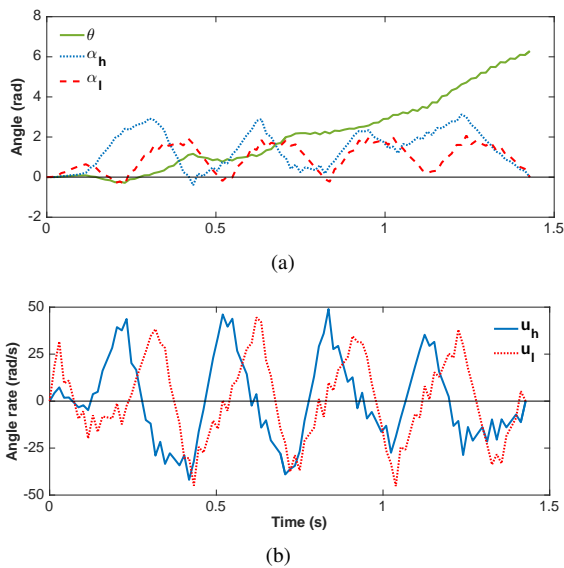


Fig. 3: Plots of the optimal (a) body orientation and shape change trajectories, (b) profiles for the drift-free case

velocities obtained through a feed-forward control input to a system with gravity that more accurately models a human diver.

### B. Drift-free Somersault

To perform a single somersault, we desire the robot to start from  $y_I = [0 \ 0 \ 0 \ 0 \ 0]$  and get to  $y_F = [2\pi \ 0 \ 0 \ 0 \ 0]$ . Assuming the robot is dropped from height of 10m, the robot has only  $\approx 1.4s$  to achieve the reorientation before hitting the ground. In this case the robot starts with zero angular momentum. The optimal flip maneuver is shown in Fig. 2, and the corresponding joint input and displacement profiles are plotted in Fig. 3.

Owing to the joint limits, the hand and leg links oscillate out-of-phase to create a geometric phase, i.e., they cause a net rotation in  $\theta$ . This behavior is very similar to the snake robot locomotion studied in [9], where the local connection

TABLE III: Physical Properties of the DiverBot

Property	Value (units)
$m_b$	45.88 kg
$m_l$	13.85 kg
$m_h$	4.45 kg
$I_b$	1.9155 kg-m <sup>2</sup>
$I_l$	0.9753 kg-m <sup>2</sup>
$I_h$	0.1579 kg-m <sup>2</sup>
$w_b$	0.553 m
$w_l$	0.837 m
$w_h$	0.66 m
$w_b^{com}$	0.2369 m
$w_l^{com}$	0.2976 m
$w_h^{com}$	0.2441 m

is due to the no-drift frictional constraints as opposed to angular momentum conservation.

### C. Two Somersaults with Initial Drift

In this case, the robot starts with some initial drift. We require the robot to go from  $y_I = [0 \ 0 \ 0 \ 0 \ 0]$  and get to  $y_F = [4\pi \ 0 \ 0 \ 0 \ 0]$ , thus accomplishing two somersaults. There is a non-zero initial body velocity,  $\dot{\theta}_0 \approx 4rad/s$ . Snapshots of the diving manoeuvre are shown in Fig. 4, while the joint and input profiles are shown in Fig. 5.

These motion profiles appear human-like. There is a compression phase that reduces the moment of inertia, then that shape is maintained for leveraging the higher rotation rate for the longest possible time, and finally they end with decompression phase to reduce the joint rates and re-orient to the desired final configuration.

### D. Discussion

All simulation results are generated on a 64-bit system using a Intel Core i7 2.66GHz machine with 8GB of RAM. Starting with a random initial seed for both cases, it still managed to find feasible and locally optimum solutions. In

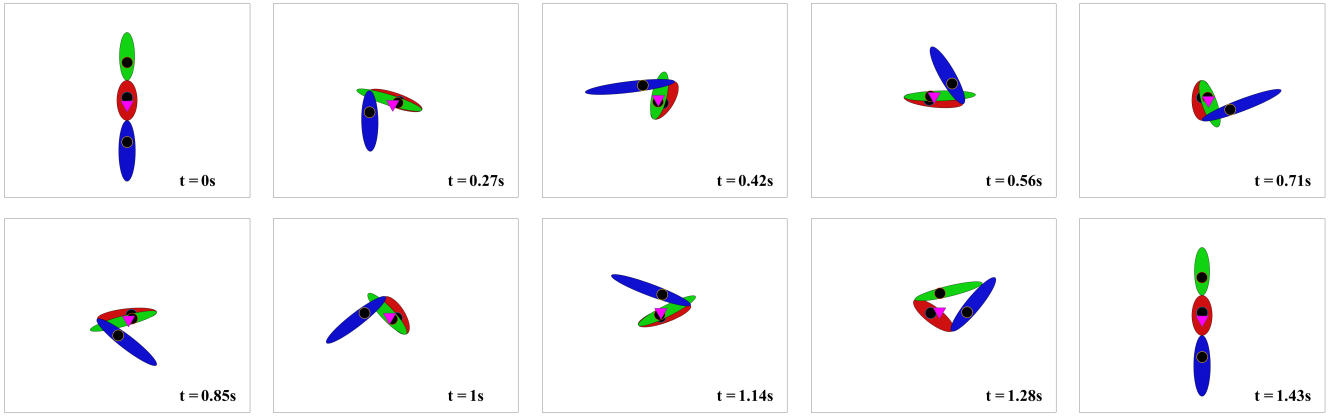


Fig. 4: Snapshots (with a time stamp) showing the DiverBot rapidly reorienting to perform a perform two somersaults when launched with initial drift in the form of non-zero body angular velocity.

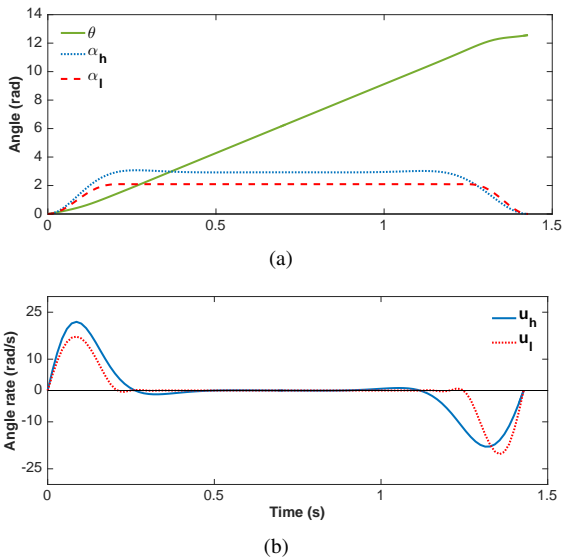


Fig. 5: Plots of the optimal (a) body orientation and shape change trajectories, (b) input profiles for the case with initial drift.

the case with drift the optimizer converged to a solution in less than a minute, while in the drift-free case convergence speed decreased and took approximately three minutes.

Overall, it can be seen that starting with a drift-free configuration requires much greater torques to perform a somersault. This is understandable as the hand and leg links, which have significantly smaller inertia than the body have to expend enough to move the body link through a complete turn in a very short time. Moreover, the local connection limits the relative effectiveness of the inputs, as can be seen from (8). On the other hand, starting with some initial drift helps the process, the leg and hand links move to only reduce the system's inertia rather than to generate the rotation. Decreasing the system inertia will consequently increase  $\dot{\theta}$  appropriately and achieve the desired reorientation within the desired time limit.

In Figs. 2 and 4, the CoM positions of the links are depicted with black circular markers, and the CoM of the DiverBot is also shown with pink triangular marker. It can be seen that during both the reorientation maneuvers, DiverBot's CoM does not go very far off the body link. The lesser its excursion, the more valid our assumption becomes of explicitly decoupling the translational and rotational dynamics of the DiverBot.

## V. CONCLUSIONS AND FUTURE WORK

In this paper, we demonstrate rapid mid-air reorientation maneuvers for the DiverBot, which is modeled after a human diver. A reduction in the dynamics was obtained from the conservation of angular momentum. Using the reduced dynamics an optimal control problem was formulated. For the DiverBot example multiple dynamically feasible joint angle input trajectories where found to perform somersaults both with and without drift. More, importantly optimal input trajectories where found in less than a minute starting from a suboptimal infeasible initial trajectory seed.

The biggest drawback for this work is performing somersaults in the absence of gravity! However, we are able to achieve the reorientations strictly within the time of flight, which atleast offers a feasible trajectory. At this point, we conjecture that any resulting variation, in the presence of gravity, would be minor and, if required, it could be corrected with a simple PD-based tracking controller. Therefore, immediate future work involves validating these ideas in the presence of gravity when  $G(q) \neq 0$ . Additionally, we intend to build a small prototype with similar mass distribution to test these optimal policies on a real robot.

## APPENDIX

The components of the DiverBot dynamics, as described in 7, are provided below:

$$\begin{aligned}
\mathbb{I}_\theta(1, 1) &= \mathbb{I}_\theta(\boldsymbol{\alpha})(2, 2) = m_h + m_b + m_l \\
\mathbb{I}_\theta(1, 2) &= \mathbb{I}_\theta(\boldsymbol{\alpha})(2, 1) = 0 \\
\mathbb{I}_\theta(1, 3) &= \mathbb{I}_\theta(\boldsymbol{\alpha})(3, 1) = m_l((w_b - w_b^{cm}) + w_l^{cm} \cos(\alpha_l)) \\
&\quad - m_h(w_b^{cm} + (w_h - w_h^{cm}) \cos(\alpha_h)) \\
\mathbb{I}_\theta(2, 3) &= \mathbb{I}_\theta(\boldsymbol{\alpha})(3, 2) = -m_l w_l^{cm} \sin(\alpha_l) \\
&\quad - m_h(w_h - w_h^{cm}) \sin(\alpha_h) \\
\mathbb{I}_\theta(3, 3) &= m_l(w_l^{cm})^2 \sin(\alpha_l)^2 + m_h(w_h - w_h^{cm})^2 \sin(\alpha_h)^2 + \\
&\quad I_h + I_b + I_l + m_h(w_b^{cm} + (w_h - w_h^{cm}) \cos(\alpha_h))^2 \\
&\quad + m_l((w_b - w_b^{cm}) + w_l^{cm} \cos(\alpha_l))^2, \\
\mathbb{I}_\alpha(1, 1) &= -m_h(w_h - w_h^{cm}) \cos(\alpha_h) \\
\mathbb{I}_\alpha(1, 2) &= -m_l w_l^{cm} \cos(\alpha_l) \\
\mathbb{I}_\alpha(2, 1) &= -m_h(w_h - w_h^{cm}) \sin(\alpha_h) \\
\mathbb{I}_\alpha(2, 2) &= m_l w_l^{cm} \sin(\alpha_l) \\
\mathbb{I}_\alpha(3, 1) &= m_h(w_h - w_h^{cm})(w_h - w_h^{cm} + w_b^{cm} \cos(\alpha_h)) \\
\mathbb{I}_\alpha(3, 2) &= -m_l w_l^{cm}(w_l^{cm} + (w_b - w_b^{cm} \cos(\alpha_l)))
\end{aligned}$$

$$\mathbf{D}_\alpha(\boldsymbol{\alpha}) = \begin{pmatrix} m_h(w_h - w_h^{cm})^2 - I_h & 0 \\ 0 & m_l(w_l^{cm})^2 + I_l \end{pmatrix}$$

$$\mathbf{C}_\alpha(1) = 2m_h w_b^{cm} \sin(\alpha_h)(w_h - w_h^{cm})(P^2/Q^2)$$

$$\mathbf{C}_\alpha(2) = 2m_l w_l^{cm} \sin(\alpha_l)(w_b - w_b^{cm})(P^2/Q^2)$$

$$\begin{aligned}
P &= (m_b m_l (w_l^{cm} + (w_b - w_b^{cm}) \cos(\alpha_l)) + \\
&\quad m_h m_l (w_l^{cm} + (w_h - w_h^{cm}) \cos(\alpha_h + \alpha_l) + w_b \cos(\alpha_l))) w_l^{cm} \dot{\alpha}_l \\
&\quad + (m_b m_l (w_h - w_h^{cm}) (-w_h - w_h^{cm}) - w_b^{cm} \cos(\alpha_h)) + \\
&\quad m_h m_l ((w_h - w_h^{cm})^2 - (w_h - w_h^{cm})(w_l^{cm} \cos(\alpha_h + \alpha_l) + w_b \cos(\alpha_h))) \dot{\alpha}_h
\end{aligned}$$

$$\begin{aligned}
Q &= m_b m_h (w_b^{cm} (w_b^{cm} + 2(w_h - w_h^{cm}) \cos(\alpha_h)) + (w_h - w_h^{cm})^2) \\
&\quad + m_b m_l (w_l^{cm} (w_l^{cm} + 2(w_b - w_b^{cm}) \cos(\alpha_l)) + (w_b - w_b^{cm})^2) \\
&\quad + m_h m_l ((w_h - w_h^{cm})^2 + 2w_l^{cm} (w_h - w_h^{cm}) \cos(\alpha_l + \alpha_h)) \\
&\quad + 2w_b (w_h - w_h^{cm}) \cos(\alpha_h) + 2w_b w_l^{cm} \cos(\alpha_l)
\end{aligned}$$

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