Differential-Flatness and Control of Multiple Quadrotors with a Payload Suspended through Flexible Cables

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Abstract—We present the coordinate-free dynamics of four different quadrotor systems with a payload suspended through flexible cables: (a) single quadrotor with a point-mass payload suspended through a flexible cable; (b) single quadrotor with tethered to the ground through a flexible cable; (c) multiple quadrotors with a shared point-mass payload suspended through flexible cables; and (d) multiple quadrotors with a shared rigid-body payload suspended through flexible cables. We model the flexible cables as a finite series of links with spherical joints with mass concentrated at the end of each link. The resulting systems are thus high-dimensional with high degrees-of-underactuation. For each of these systems, we show that the dynamics are differentially-flat, enabling planning of dynamically feasible trajectories. For the single quadrotor with a point-mass payload suspended through a flexible cable with five links (16 degrees-of-freedom and 12 degrees-of-underactuation), we use the coordinate-free dynamics to develop a geometric variation-based linearized equations of motion about a desired trajectory. We show that a finite horizon linear quadratic regulator can be used to track a desired trajectory with a relatively large region of attraction.

I. INTRODUCTION

Aerial transportation through small unmanned aerial vehicles (UAVs) has shown great potential in recent years, especially with the commercialization of UAV-based package and mail delivery. Consequently, the automatic control of quadrotors to transport payloads has been the focus for many research groups. Load carrying using quadrotor UAVs can be realized either by rigidly attaching the load to the quadrotor or through the suspension of cables. Rigid attachment of the load can be achieved by attaching the load to the quadrotor’s fuselage or by onboard grippers [11, 20]. This type of rigid attachment makes control design much easier for load transportation and grasping through a single quadrotor. However, both of these methods increase the inertia of the quadrotor, making it sluggish for fast attitude response and agile disturbance rejection. This could potentially reduce the robustness of the overall system.

To overcome this shortcoming, an alternative method has been proposed where the payload is suspended through cables. A cable-suspended load system increases the degrees of underactuation, making planning and control for such systems more challenging. Early work on cable-suspended load transportation systems have been studied for helicopters [2], [1]. Control of UAVs with suspended load have been addressed through trajectory generation of fast load transport with minimized swing [14], [23], [16], or through modeling the suspended load as an external disturbance and developing robust controllers to reject these disturbances [15]. Geometric control design has been developed in [18], [19] to track a smooth aggressive trajectory. Similar geometric controllers have been proposed in [21], which allows the load to undergo large swings. Similar controllers for suspended loads have been developed in [17], [9], [21], where the load is supported from multiple quadrotors.

However, these controllers assume that the suspended cable is massless and that the cable is always taut. In particular, they do not address the control challenges when the cable is not taut or when the cable is deformed. These assumptions may not hold in reality, especially when the mass of the cable is not zero and distributed or in cases where the tension in the cable is very small. In this case, the stability of these controllers would get worse, and thus the mass distribution of the cable needs to be considered in the dynamics. However, a continuous mass distribution would result in a configuration space of infinite dimension with the dynamics being represented through partial differential equations. To reduce the modeling complexity, a general methodology is to employ a finite element approximation for the cable, where the cable is approximated as a series of links connected by spherical joints [4, 5, 7]. Goodarzi et al. [4] first develops dynamics of a single quadrotor transporting a point-mass through a flexible cable. Based on this, [6] extends to the case of a rigid body load with multiple quadrotors. Although both of these work present a coordinate-free model and use linearization for regulation control, the resulting controller can only stabilize to a setpoint corresponding to the quadrotor hovering and the payload suspended vertically. In particular, the developed
controller is unable to track a desired trajectory.

In this paper, we focus on the properties of four particular transportation systems with flexible cables. With respect to the prior work in [4, 5] which proposes regulation control of the load’s pose, our aim is to investigate further into the planning and tracking of desired dynamically feasible trajectories for such systems. The contributions of this paper with respect to prior work is as follows:

- We develop coordinate-free dynamics of four different quadrotor systems with a payload suspended through flexible cables using the Newton-Euler method. We prove that the resulting dynamics for these systems are differentially-flat and provide flat outputs candidates.
- For the single quadrotor with point-mass payload suspended through a flexible cable, we present a geometric variation-based linearization of the system dynamics with respect to a desired reference trajectory.
- We use the linearized dynamics to develop a finite-horizon linear quadratic regulator and demonstrate trajectory tracking on the nonlinear system to achieve load trajectory tracking of any sufficiently smooth trajectory. We demonstrate the large region of attraction of the controller through numerical simulations.

The rest of the paper is organized as follows. Section II presents the dynamical models for a single or multiple quadrotor(s) with a payload suspended through flexible cables, where an individual flexible cable is modeled as a series of \( n \) small links as illustrated in Fig. 1–Fig. 4. We describe the coordinate-free dynamics for these four systems based on the rotation matrix \( SO(3) := \{R \in \mathbb{R}^{3 \times 3} | R^T R = I, det(R) = +1 \} \) for quadrotor attitude, and the two-sphere \( S^2 := \{ q \in \mathbb{R}^3 | q.q = 1 \} \) for each of the \( n \) links, as segments of the cable.

The configuration of the systems under consideration can be given by the pose of the load in inertial frame(position for point-mass load and position and orientation of the rigid-body load), attitudes of each link in the flexible cable(s) and the attitude of the quadrotor(s). Equations of motion can be derived using Lagrange-d’Alembert’s principle, which states that the variation of the action integral is equal to the negative of virtual work done by the external forces,

\[
\delta \int_{t_0}^{t_1} \mathcal{L} dt + \int_{t_0}^{t_1} \left( (W_1, M) + W_2, f R e_3 \right) dt = 0, \tag{1}
\]

where \( \mathcal{L} \) is the corresponding Lagrangian, which equals kinetic energy minus potential energy.

Fig. 2: Quadrotor with a point-mass payload suspended through a flexible cable. The flexible cable is modeled as a series of links connected by \( S^2 \) joints. The system evolves on \( SO(3) \times \mathbb{R}^3 \times (S^2)^n \) and has \((6 + 2n)\) degrees of freedom with \((2 + 2n)\) degrees of underactuation.

A detailed description on the derivation method can be found in [8]. As proposed, this type of representation can result in a more compact equations of motion on the nonlinear manifolds. However, we would utilize a different version which makes it more convenient for further derivation.

### A. Quadrotor with load suspended through a flexible cable (Fig. 2)

The first system is a single quadrotor with load suspended through a flexible cable shown in Fig. 2. This flexible cable is modeled as a chain of \( n \) links, and the suspended load is considered to be a point mass at the end of the \( n^{th} \) link. The configuration space is given by \( Q = SO(3) \times \mathbb{R}^3 \times (S^2)^n \), where \( n \) is the number of links in the cable. In the finite element approximation, we assume that the mass of the smaller links is concentrated at the end of each link. The relation between the different link-mass positions, quadrotor center of mass and load is given as

\[
x_i = x_{i-1} + l_i q_i, \tag{2}
\]

where \( i \in \{1, 2, \ldots, n\} \) and \( l_i, q_i, x_i \) are the length, the unit directional vector and the position of the \( i^{th} \) link; \( x_0 \) and \( x_n \) are the positions of quadrotor center of mass and load; \( R \) and \( \Omega \) are the rotation matrix of the quadrotor and its body-fixed angular velocity. Let \( T_j \in \mathbb{R} \) be the magnitude of the tension in the \( j^{th} \) link, the dynamics of the system with \( n \) link flexible cable can be written as follows,

\[
m_Q (\ddot{x}_0 + g e_3) = f R e_3 + T_1 q_1, \tag{3}
\]

\[
m_j (\ddot{x}_j + g e_3) = - T_j q_j + T_{(j+1)} q_{(j+1)}, \tag{4}
\]

\[
m_n (\ddot{x}_n + g e_3) = - T_n q_n, \tag{5}
\]

\[
J \ddot{\Omega} + \dot{\Omega} J \Omega = M, \tag{6}
\]

for all \( j \in \{1, 2, \ldots, (n - 1)\} \) where \( m_Q, J, f \) and \( M \) are the mass, inertia matrix, thrust and the moment represented in body frame of the quadrotor, \( m_i \) is the mass of the \( i^{th} \) link.

Remark 1. This system has \( 6 + 2n \) degrees of freedom, with 4 degree of actuation from the thrust and moment \((f, M)\). Thus the degree of underactuations for the system is \((2n + 2)\)
TABLE I: Various symbols used in defining the Dynamics of different systems

<table>
<thead>
<tr>
<th>Flexible suspended load and tethered quadrotor</th>
<th>Explanation</th>
<th>Multiple quadrotors with suspended point and rigid body load wet $i^{th}$ quadrotor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \quad n \in \mathbb{R}$</td>
<td>Number of quadrotors</td>
<td>$p \in \mathbb{R}$</td>
</tr>
<tr>
<td>$m_Q \in \mathbb{R}$</td>
<td>Mass of the quadrotor</td>
<td>$m_i \in \mathbb{R}$</td>
</tr>
<tr>
<td>$J \in \mathbb{R}^{3 \times 3}$</td>
<td>Inertia matrix of the quadrotor</td>
<td>$J_i \in \mathbb{R}^{3 \times 3}$</td>
</tr>
<tr>
<td>$R \in SO(3)$</td>
<td>Rotation matrix of the quadrotor from body-fixed frame to inertial frame</td>
<td>$R_i \in SO(3)$</td>
</tr>
<tr>
<td>$\Omega \in \mathbb{R}^3$</td>
<td>Body-frame angular velocity</td>
<td>$\Omega_i \in \mathbb{R}^3$</td>
</tr>
<tr>
<td>$x_0, v_0 \in \mathbb{R}^3$</td>
<td>Position and velocity vectors of quadrotor’s center of mass in the inertial frame</td>
<td>$x_{0i}, v_{0i} \in \mathbb{R}^3$</td>
</tr>
<tr>
<td>$m_n \in \mathbb{R}$</td>
<td>Mass of the suspended Load</td>
<td>$m_L \in \mathbb{R}$</td>
</tr>
<tr>
<td>$x_n, v_n \in \mathbb{R}^3$</td>
<td>Position and velocity vectors of load position in the inertial frame</td>
<td>$x_{Li}, v_{Li} \in \mathbb{R}^3$</td>
</tr>
<tr>
<td>$q_j \in S^2$</td>
<td>Attitude of the $j^{th}$ link i.e., unit vector representing the link orientation in the inertial frame</td>
<td>$q_{ji} \in S^2$</td>
</tr>
<tr>
<td>$\omega_j \in \mathbb{R}^3$</td>
<td>Angular velocity of the $j^{th}$ link wrt. the interal frame</td>
<td>$\omega_{ji} \in \mathbb{R}^3$</td>
</tr>
<tr>
<td>$m_j \in \mathbb{R}$</td>
<td>Mass of the $j^{th}$ link, concentrated at the end of the link</td>
<td>$m_{ji} \in \mathbb{R}$</td>
</tr>
<tr>
<td>$x_j, v_j \in \mathbb{R}^3$</td>
<td>Position and Velocity of the mass $m_j$ in the inertial frame</td>
<td>$x_{ji}, v_{ji} \in \mathbb{R}^3$</td>
</tr>
<tr>
<td>$l_j \in \mathbb{R}$</td>
<td>Length of the $j^{th}$ link</td>
<td>$l_{ji} \in \mathbb{R}$</td>
</tr>
<tr>
<td>$T_j$</td>
<td>Tension vector of the $j^{th}$ link</td>
<td>$T_{ji}$</td>
</tr>
<tr>
<td>$T_j$</td>
<td>Magnitude of tension in $j^{th}$ link</td>
<td>$T_{ji}$</td>
</tr>
<tr>
<td>$f \in \mathbb{R}$</td>
<td>Magnitude of the thrust of the quadrotor</td>
<td>$f_i \in \mathbb{R}$</td>
</tr>
<tr>
<td>$M \in \mathbb{R}^3$</td>
<td>Moment vector of the quadrotor in the body-fixed frame</td>
<td>$M_i \in \mathbb{R}^3$</td>
</tr>
<tr>
<td>$e_1, e_2, e_3 \in \mathbb{R}^3$</td>
<td>Unit vectors along the x,y,z directions of the inertial frame</td>
<td>$e_1, e_2, e_3 \in \mathbb{R}^3$</td>
</tr>
<tr>
<td>$fRe_3 \in \mathbb{R}^3$</td>
<td>is the thrust generated by the quadrotor in the inertial frame</td>
<td>$f_iR_i e_3 \in \mathbb{R}^3$</td>
</tr>
</tbody>
</table>

Remark 2. The assumption that the flexible cable is a series of connected links may not be valid under some extreme conditions. However, this offers more flexibility over the assumption of a single massless link and can be potentially used to design more aggressive trajectories that require cable deformation.

B. Quadrotor tethered using a flexible cable (Fig. 3)

The second system is a quadrotor tethered to a fixed base using a flexible cable, whereas the end of the $n^{th}$ link cable is tethered to the base. Let $x_n$ be the position of this fixed point and $\vec{T}_n$ be the tension vector in the $n^{th}$ link. The relation between the position of different links is as given in Eq. 2 where $\vec{T}_n = T_n q_n$, $T_n$ is magnitude of the Tension and $q_n$ is the attitude of the $n^{th}$ link. The corresponding dynamics is shown below:

$$m_Q(x_0 + ge_3) = fRe_3 + T_1 q_1,$$

$$m_j(x_j + ge_3) = -T_j q_j + T_{j+1} q_{j+1},$$

$$J\dot{\Omega} + \Omega J\dot{\Omega} = M,$$

where $j \in \{1, 2, \ldots, (n-1)\}$.

Remark 3. Tethered quadrotor using flexible cable of $n$ links is a $2n + 3$ DOF system, with 4 degrees of actuation. Thus the degree of underactuations for the system is $(2n - 1)$. If $n$ is zero, there is no cable and the center of mass of the quadrotor is fixed to the base.

![Fig. 3: Quadrotor tethered to a fixed ground using a flexible cable. The flexible cable is modeled as a chain of $n$ links with the $n^{th}$ link tethered to a fixed ground. The configuration space is $SO(3) \times (S^2)^n$ with $(3+2n)$ degree of freedom and $(2n-1)$ degrees of underactuation.](image)

C. Point-mass load suspended from multiple quadrotors through flexible cables (Fig. 4)

The third system is a point mass load suspended by $p$ quadrotors through flexible cables as shown in Fig. 1 where $p > 1$. The configuration variables are the load position $x_L \in \mathbb{R}^3$, attitude of of each link in the flexible cable.
Fig. 4: Multiple quadrotors with a shared rigid-body payload suspended through flexible cables.

$q_{ij} \in S^2$ (here, $q_{ij}$ corresponds to the attitude of the $j^{th}$ link in the flexible cable of the $i^{th}$ quadrotor) and attitude of the quadrotors $R_i \in SO(3)$. Tab. I explains different notations followed in this paper. Positions of different links $x_{ij}$ and quadrotors $x_{i0}$ can be obtained from the kinematic relations below,

$$x_L = x_{i(n_i-1)} + l_{i n_i} q_{in_i}$$

where $j \in \{1, \ldots, (n_i-1)\}$, $i \in \{1, 2, \ldots, p\}$, and $n_i$ is the number of links in the $i^{th}$ flexible cable.

Multiple quadrotors and multiple links in each cable result in a complicated system with high degree of underactuations. The configuration space of this system is given as $Q = \mathbb{R}^3 \times \prod_{i=1}^p (SO(3) \times (S^2)^{n_i})$. The corresponding system dynamics can be described in terms of internal tensions $T_{ij} > 0$ shown below,

$$m_i(\ddot{x}_{i0} + ge_3) = f_i R_i e_3 + T_{i1} q_{i1}$$

$$m_{ij}(\ddot{x}_{ij} + ge_3) = -T_{ij} q_{ij} + T_{i(j+1)} q_{i(j+1)}$$

$$m_L(\ddot{x}_L + ge_3) = -\sum_{i=1}^n T_{in_i} q_{in_i}$$

$$J_{i} \ddot{\Omega}_i + \Omega_i J_i \ddot{\Omega}_i = M_i$$

for $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, (n_i-1)\}$ where $m_L$ is mass of the load and $m_i$, $J_i$, $f_i$ & $R_i$ are the mass, inertia matrix, thrust and rotation matrix of the $i^{th}$ quadrotor.

Remark 4. Degrees of freedom in the system is $(3 + 3p + 2 \sum_{i=1}^p n_i)$ and has $4p$ actuators. Thus, underactuation in the system is $(3 + p + 2 \sum_{i=1}^p n_i)$

D. Rigid body load suspended from multiple quadrotors through flexible cables (shown in Fig. 4)

We now consider the last system, where a rigid body payload with mass $m_L$, inertia matrix $J_L$ and orientation of $R_L$ in inertial frame, is suspended by $p$ quadrotors through flexible cables. Figure 4 shows the geometry of the rigid load suspended from $p$ quadrotors by flexible cables. Kinematic relation between different positions of cable links, quadrotors’ and load position are given as follows,

$$x_{i(n_i-1)} = x_L + R_L r_i - l_{in_i} q_{in_i}$$

$$x_{i(j-1)} = x_{ij} - l_{ij} q_{ij}$$

for $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, (n_i-1)\}$ where $n_i$ is the number of links in the i$^{th}$ flexible cable, $x_L$ is the center of mass’ position of the load and $x_{i0}$ is the center of mass’ position of the i$^{th}$ quadrotor.

For the case of a rigid body load, the degrees of freedom is increased by 3 in the attitude, compared to the case of point mass load. Similar to the previous system, we can use the internal tensions $T_{ij}$ on the corresponding configuration space $SE(3) \times \prod_{i=1}^p (SO(3) \times (S^2)^{n_i})$ to express the dynamics shown below,

$$m_i(\ddot{x}_{i0} = f_i R_i e_3 - m_i g e_3 + T_{i1} q_{i1}$$

$$J_{i} \ddot{\Omega}_i + \Omega_i J_i \ddot{\Omega}_i = M_i$$

$$m_{ij}(\ddot{x}_{ij} + ge_3 = -T_{ij} q_{ij} + T_{i(j+1)} q_{i(j+1)}$$

$$m_L(\ddot{x}_L + ge_3 = -\sum_{i=1}^n T_{in_i} q_{in_i} - m_L g e_3$$

$$J_L \ddot{\Omega}_L + \Omega_L J_L \ddot{\Omega}_L = -\sum_{i=1}^n (r_i \times R_i^T T_{in_i} q_{in_i})$$

for $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, (n_i-1)\}$ and all other symbols with similar representation as in previous subsection.

Remark 5. Degrees of freedom for the rigid body load suspended from multiple quadrotors through flexible cables is $(6+3p+2 \sum_{i=1}^p n_i)$ and has $4p$ actuators. Thus, underactuation in the system is $(6 + p + 2 \sum_{i=1}^p n_i)$

To better distinguish different symbols used, we have provided a table shown in Tab. I.

III. DIFFERENTIAL FLATNESS

Differential flatness is the property of nonlinear systems, which identifies certain flat outputs for the system, such that all the system states and the inputs can be expressed as smooth functions of flat outputs and a finite number of their derivatives. In this section, we show that the four quadrotor systems described previously are differentially flat.

Differential flat systems have properties which can be used for feedback linearization. This concept has been previously exploited, to develop trajectories and achieve trajectory tracking control for loads suspended from quadrotors in [10, 18] and [19]. According to [13], differential flatness is defined as,

Definition 1. Differentially-Flat System [13]: A system $\dot{x} = f(x, u)$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, is differentially flat if there exists flat outputs $y \in \mathbb{R}^m$ of the form $y = y(x, u, \dot{u}, \ldots, u^{(p)})$ such that the states and the inputs can be expressed as $x = x(y, \dot{y}, \ldots, y^{(p)})$, $u = u(y, \dot{y}, \ldots, y^{(q)})$, where $p, q$ are nonnegative integers.

The following subsections show the differential flatness for different systems described earlier separately.

A. Quadrotor with load suspended through a flexible cable

Lemma 1. Differential flatness of quadrotor with payload suspended through a flexible cable: $\mathcal{Y} = (x_n, \psi)$ are the set of flat-outputs for the quadrotor with point mass load suspended through flexible cables, where $x_n \in \mathbb{R}^3$ is the position of the load (the n$^{th}$ point mass) and $\psi \in \mathbb{R}$ is the yaw angle of the quadrotor.
TABLE II: Comparison between different quadrotor systems with suspended load

<table>
<thead>
<tr>
<th>No. of quadrotors</th>
<th>Quadrotor with flexible cable suspended load</th>
<th>Quadrotor tethered with a flexible cable</th>
<th>Point mass load suspended by multiple quadrotors</th>
<th>Rigid body load suspended by multiple quadrotors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Independent DOF</td>
<td>$x_{n} \in \mathbb{R}^{4}$</td>
<td>$q_{1}, q_{2}, \ldots, q_{n} \in \mathbb{S}^{2}$</td>
<td>$x_{L} \in \mathbb{R}^{3}$</td>
<td>$(x_{L}, R_{L}) \in \mathbb{S}E(3)$</td>
</tr>
<tr>
<td>No. of DOF</td>
<td>$6 + 2n$</td>
<td>$3 + 2n$</td>
<td>$3 + 3p + 2 \sum_{i=1}^{n} p_{i}$</td>
<td>$6 + 3p + 2 \sum_{i=1}^{n} p_{i}$</td>
</tr>
<tr>
<td>No. of Actuators</td>
<td>$2 + 2n$</td>
<td>$4$</td>
<td>$4p$</td>
<td>$4p$</td>
</tr>
<tr>
<td>Flat outputs</td>
<td>$\mathcal{Y} = \begin{bmatrix} x_{n} \in \mathbb{R}^{4} \ \psi \in \mathbb{R} \end{bmatrix}$</td>
<td>$\mathcal{Y} = \begin{bmatrix} x_{L} \in \mathbb{R}^{3} \ \psi_{L} \in \mathbb{R} \end{bmatrix}$</td>
<td>$\mathcal{Y} = \begin{bmatrix} x_{L} \in \mathbb{R}^{3} \ \psi_{k} \in \mathbb{R} \end{bmatrix}$</td>
<td>$\mathcal{Y} = \begin{bmatrix} R_{L} \in \mathbb{S}O(3) \ \Lambda \in \mathbb{R}^{3n-6} \end{bmatrix}$</td>
</tr>
<tr>
<td>No. of flat outputs</td>
<td>$4$</td>
<td>$7$</td>
<td>$4p$</td>
<td>$4p$</td>
</tr>
</tbody>
</table>

\[ \mathcal{T}_{n} = T_{n}q_{n} \] can be calculated from (5), since the quantity $\hat{x}_{n}$ is known from the flat-output $x_{n}$. Unit vector along the $n^{th}$ link and magnitude of the tension $T_{n}$ can be determined as $q_{n} = (T_{n}q_{n})/||T_{n}q_{n}||$ and $T_{n} = (T_{n}q_{n})q_{n}$. Tensions in all the remaining $(n-1)$ links can be calculated from (4) iteratively. Positions of all other links and quadrotor position can be determined from (2). Since $(x_{0}, \psi)$ are the flat-outputs of a quadrotor (10), the rest of the states $(R, \Omega)$ and inputs $(f, M)$ can be calculated.

Remark 6. To completely define and calculate all the states and inputs of the above system with $n-$ chain links, requires $(2n + 4)$ derivatives of the flat-output $x_{n}$ and $2^{nd}$ derivative of the yaw angle $\psi$.

B. Quadrotor tethered using a flexible cable

**Lemma 2.** Differential flatness of a tethered quadrotor using a flexible cable: $\mathcal{Y} = (x_{n}, \hat{T}_{n}, \psi)$ are the set of flat-outputs for tethered quadrotor using flexible cable, where $x_{n}$ is position of the fixed position of the tether, $\hat{T}_{n} = T_{n}q_{n} \in \mathbb{R}^{3}$ is the tension vector in $n^{th}$ link and $\psi \in \mathbb{R}$ is the yaw angle of quadrotor

**Proof:** Magnitude and orientation of tension in the $n^{th}$ link, can be calculated from the flat output as, $T_{n} = || \hat{T}_{n} ||$ and $q_{n} = \hat{T}_{n}/T_{n}$. Since $x_{n}$ is a fixed point, its higher derivatives are zero. Thus, knowing $x_{n}$ and $q_{n}$, we can calculate the rest of the positions using (2), (5) gives the values for tensions in the remaining links. The rest follows from Lemma 1 which shows that this system is differentially-flat.

Remark 7. Tethered quadrotor with flexible cable of $n - 1$ links, requires $(2n + 2)$ derivatives of the flat-output $\hat{T}_{n}$ and the $2^{nd}$ derivative of the yaw angle $\psi$.

**Corollary 1.** $\mathcal{Y} = (x_{n}, \psi, \hat{F})$ are the flat-outputs for quadrotor with point mass load suspended via flexible cable with an external force $(\hat{F})$ acting on the point mass load, where $x_{n} \in \mathbb{R}^{3}$ is the position of the load $(n^{th}$ point mass) and $\psi \in \mathbb{R}$ is the yaw angle of the quadrotor.

C. Point mass load suspended from multiple quadrotors through flexible cables

**Lemma 3.** Differential flatness of the system, with point mass suspended by $(p \geq 1)$ quadrotors via flexible cables with $(n_{i} \geq 1)$ links for $i \in \{1, \ldots, n\}$: $\mathcal{Y}_{n} = (x_{L}, \hat{T}_{n}, q_{in}, \psi_{j})$ for $i \in \{2, \ldots, p\}$ and $j \in \{1, \ldots, p\}$ are the flat-outputs for the given system, with $T_{n_{i}} = T_{n_{i}}q_{in_{i}} \in \mathbb{R}^{3}$ the tension in the last link of $(p-1)$ cables and $\psi_{j} \in \mathbb{R}$ the yaw angle of the quadrotors. $x_{L} \in \mathbb{R}^{3}$ is the load position.

**Proof:** From flat-output $x_{L}$ and its higher derivatives we can calculate $\sum_{i} T_{n_{i}}, q_{in_{i}}$, from Eq. (14). Knowing the values of $T_{n_{i}}, q_{in_{i}}$, and its higher derivatives for $i \in \{2, \ldots, p\}$ we can calculate the value of $T_{n_{i}}, q_{in_{i}}$, and its higher derivatives. Positions of different links of the systems can be calculated from Eq. (10)-Eq. (11). Thus, we know the positions and tensions of the last link and their derivatives. The rest of the proof follows from Lemma 1.

Remark 8. To completely describe all states and inputs of the above system as a function of flat-outputs required, we need to differentiate $x_{L}$ $(4 + 2 \max\{n_{1}, \ldots, n_{p}\}$) times, $T_{n_{i}}, q_{in_{i}}$ $(2 + 2n_{i})$ times for $i \in \{2, \ldots, p\}$ and $\psi_{j}$ twice.

D. Rigid body load suspended from multiple quadrotor through flexible cables

**Lemma 4.** Differential flatness of the system, with rigid body payload suspended by $(p \geq 1)$ quadrotors via flexible cables with $(n_{i} \geq 1)$ links for $i \in \{1, \ldots, n\}$: $\mathcal{Y}_{n} = (x_{L}, R_{L}, \Lambda, \psi_{j})$ for $j \in \{1, \ldots, p\}$ is a set of flat outputs for the given system, where $\Lambda \in \mathbb{R}^{3n-6}$ satisfies,

\[ \mathcal{T} = \Phi^{\dagger}W + NA \]  

with $T, W$ defined as

\[ T = \begin{bmatrix} R_{L}^{T}T_{n_{1}}, q_{in_{1}} \\ R_{L}^{T}T_{n_{2}}, q_{in_{2}} \\ \vdots \\ R_{L}^{T}T_{n_{p}}, q_{in_{p}} \end{bmatrix}, W = - \begin{bmatrix} R_{L}^{T}m_{L}(\hat{\chi}_{L} + \omega_{o}\omega_{L}) \\ J_{L}\Omega_{L} + \Omega_{L}J_{L}\Omega_{L} \end{bmatrix} \]

and $\Phi^{\dagger}$, $N$ are respectively the Moore-Penrose generalized inverse and the nullspace of

\[ \Phi = \begin{bmatrix} I & I & \ldots & I \\ \hat{r}_{1} & \hat{r}_{2} & \ldots & \hat{r}_{n} \end{bmatrix} \]  

(25)
provided that both $\Phi^i$ and $N$ exist.

Proof: From Eq. (21) and Eq. (22), we get,

\[
\begin{bmatrix}
R^T_1 T_{11n_1} q_{1n_1} \\
R^T_1 T_{21n_2} q_{2n_2} \\
\vdots \\
R^T_1 T_{nn_n} q_{nn_n}
\end{bmatrix} = \Phi
\]

(26)

Proof follows from [Lemma 2, 17], where the tensions for the last links of each flexible cable can be calculated. To compute the higher order derivatives of these tensions, we need to make sure that the time-invariant matrices $\Phi^i$ and $N$ exist. Positions for links of cable can be calculated from Eq. (16) and Eq. (17). Knowing position and tensions in the last link for each cable, from Lemma 11, we can calculate the rest of the states can be calculated.

Remark 9. Calculation of all the states and inputs for the system requires, up to $2^{nd}$ derivative of $\psi_j$, $(2 + 2n_{max})$ (where, $n_{max} = \max\{n_1, \ldots, n_p\}$) derivates of $T$, which in turn depends on $W$ & $\Lambda$. Thus, requires $(2 + 2n_{max})$ derivates of $\Lambda$ and $(4 + 2n_{max})$ derivates of $x_L$ & $R_L$.

So far we have discussed about differential flatness in different quadrotor-load with flexible cable systems. In Table II, we present the comparison between four different differential flat systems. In the next section, we linearize the dynamics of load suspended from quadrotor using flexible cable about a specific desired time varying trajectory. This linearization is performed directly on the manifolds and thus is singularity free.

IV. CONTROL DESIGN OF A SINGLE QUADROTOR WITH POINT-MASS LOAD SUSPENDED THROUGH A FLEXIBLE CABLE

We have previously shown the equations of motion for quadrotor with a load suspended through a flexible cable in Eq. (3), Eq. (8). Designing a controller based on this model is not feasible since the values of each tension vectors remain unknown. Thus for the purpose of control, we use instead the compact geometric equations of motion developed in [4]. The system dynamics are linearized about a time-varying reference trajectory to obtain a linear time-varying dynamics. However, since the system evolves on a complex manifold as the configuration space $Q = \mathbb{R}^3 \times SO(3) \times (S^2)^n$, standard linearization techniques is cumbersome to implement or involves complex calculations using local variables and can also result in singularities. Variation based geometric linearization is used to overcome these difficulties. We refer to [22] for detailed discussion.

A. Variation Expressions

The distance between points on a manifold can be measured through the concept of configuration error. The infinitesimal variations can be considered as a linear approximation of this configuration error on the manifold. Geometric linearization is to get the dynamics of the infinitesimal variations in the form of linear system. For the purpose of control, we could roughly treat the variation as the error between the planned trajectory and the actual state. The corresponding expressions on $\mathbb{R}^3$, $S^2$ and $SO(3)$ are given as follows.

Remark 10. The subscript ‘d’ in the rest of the section refers to the time varying desired reference trajectory. For a given sufficiently smooth load trajectory profile, reference trajectory for all the states can be calculated, since the system is differentially flat.

1) Variation in $\mathbb{R}^3$: Infinitesimal variation in Cartesian space $\mathbb{R}^3$ with respect to a reference position vector $x_d(t) \in \mathbb{R}^3$ and velocity $v_d(t) \in \mathbb{R}^3$ are,

\[
\delta x(t) = x(t) - x_d(t), \quad \delta v(t) = v(t) - v_d(t)
\]

For such flat space, the linear error state in $\mathbb{R}^3$ is the exact distance as,

\[
\begin{bmatrix}
\delta x \\
\delta v
\end{bmatrix} = \begin{bmatrix}
x(t) - x_d(t) \\
v(t) - v_d(t)
\end{bmatrix} \equiv \delta s(t)
\]

(27)

2) Variation in $S^2$: Infinitesimal variation in $S^2$ with respect to a desired unit direction vector $q_d(t) \in S^2$ can be calculated as,

\[
\delta q(t) = \frac{d}{dt} e(\xi) q_d(t) = \xi \times q_d(t)
\]

subject to the constraints,

\[
\xi \cdot q_d = 0, \quad \delta \omega \cdot q_d + \omega_{d} \cdot (\xi \times q_d) = 0
\]

If the actual direction $q(t)$ is close to the desired direction vector $q_d(t)$, we can approximate the linear error states $[\xi, \delta \omega]$ to the actual error calculated previously. Then the corresponding linear error on $S^2$ manifold is given as,

\[
\begin{bmatrix}
\delta \xi \\
\delta \omega
\end{bmatrix} \approx \begin{bmatrix}
e(\xi) e(\omega_{d}) q_d(t) \\
e(\xi) e(\omega_{d}) (\omega(t) - (q_d(t) \omega_{d}(t)))
\end{bmatrix}
\]

The configuration error for the cable link’s direction on $S^2$ is given as,

\[
\Psi_{q_i} = (1 - q_i, q_{id})
\]

(30)

3) Variation in $SO(3)$: Infinitesimal variation in $SO(3)$ with respect to a desired rotation matrix $R_d(t) \in SO(3)$ can be calculated as given in [22],

\[
\delta R(t) = \frac{d}{dt} e(\eta) = R_d(t) \tilde{\eta},
\]

(31)

where $\eta \in \mathbb{R}^3$. In a similar manner, the infinitesimal variation of body-angular velocities is given as,

\[
\delta \Omega(t) = \tilde{\Omega}(t) \eta(t) + \dot{\eta}(t).
\]

(32)

If the actual rotation matrix $R(t)$ is close to the desired rotation matrix $R_d(t)$, it can be assumed that $[\eta, \tilde{\Omega}]$ are linear approximation of the error $[e_R, e_{\Omega}]$ between the actual and desired rotation matrices and angular velocities. Then we denote the error state as,

\[
\begin{bmatrix}
\eta \\
\delta \Omega
\end{bmatrix} \approx \begin{bmatrix}
e_R \\
e_{\Omega}
\end{bmatrix} = \frac{1}{2} \left( R^T_R(t) R(t) - R^T(t) R_d(t) \right) \Omega_d(t) \Omega_d(t)^T
\]

(33)

The configuration error for the quadrotor rotation matrix on $SO(3)$ is given below,

\[
\Psi_R = \frac{1}{2} (trace(I - R^T_R R))
\]

(34)

B. Linearized Dynamics

Equations of motion given in Eq. (5) – (6) is converted to a new compact representation in [4]. Based on this compact representation, we are able to linearize it about
a desired trajectory using the variation techniques described previously. We list all the error states of the system as \( \{ \eta, \delta \Omega, \delta x_0, \xi_1, \ldots, \xi_n, \delta \omega_0, \omega_1, \ldots, \delta \omega_n \} \), where \((\xi_i, \delta \omega_i)\) correspond to the linear error state approximation for the direction vector of the \(i^{th}\) link.

The linearized dynamics are given in the following equations:

\[
\dot{s} = As + B\delta u, \\
Cs = 0,
\]

where the state and the input are,

\[
s = \begin{bmatrix} \eta & \delta \Omega & \delta x_0 & \xi_1 & \ldots & \xi_n \end{bmatrix}^T \\
\delta u = \begin{bmatrix} \delta f & \delta M \end{bmatrix}^T
\]

where the expressions of \(A, B, C\) are omitted here due to space constraints.

C. Finite Horizon Linear Quadratic Regulator (LQR)

Note that the resulting linearized equation \((35)\) is essentially a time-varying linear system. Thus, any standard control techniques used for a linear system are applicable. Since the system is time-varying, we implement a finite LQR controller. A reference trajectory \(x_d\) is considered along with the feedforward input \(u_d = [f_d, M_d]^T\) to achieve this reference trajectory. The state \(s(t)\) gives the error in the system, which can be calculated using \((35), (29)\) and \((27)\) in IV-A.

A finite horizon \(T\) is chosen along with the positive semi-definite matrices \(Q_1 = Q_1^T \geq 0 \in \mathbb{R}^{n \times n}\) and \(Q_2 = Q_2^T \geq 0 \in \mathbb{R}^{m \times m}\), where \(Q_1\) and \(Q_2\) are weight matrices corresponding to the states \(s(t)\) and control inputs \(\delta u\). Here, \(n\) is equal to the number of states and \(m\) is the number of inputs. We also choose the final weight matrix for at \(T\), \(P(T) = P_T = P_T^T \geq 0 \in \mathbb{R}^{n \times n}\) as the weight matrix for the terminal state \(s(T)\).

To solve for the optimal solution of the finite-horizon LQR, we need to first solve the continuous-time Riccati equation given below,

\[
\dot{P}(t) = Q_1 - P(t)B(t)Q_2^{-1}B(t)^TP(t) + A(t)^TP(t) + P(t)A(t).
\]

For real-time implementation, we need to integrate Eq. \((39)\) backwards in time from \(t = T\) to \(t = 0\), with the terminal condition \(P(T) = P_T\). The precomputed values of \(P(t)\) are stored in a table for calculating the feedback gain online. Then the value \(P(t)\) is used to calculate feedback control input for the linear system \((35)\) as,

\[
\delta u(t) = -K(t)s(t) = -Q_2^{-1}B(t)P(t)s(t),
\]

Finally, the trajectory tracking controller can be calculated as,

\[
u(t) = u_d(t) + \delta u(t),
\]

where \(u(t) = [f(t), M(t)]^T\), and the gain matrix \(K(t)\) can be computed online based on the stored values of \(P(t), B(t)\) and \(Q_2\).

V. SIMULATION RESULTS

Based on the controller in previous section, we are able to test it out in simulation. In particular, we are using Matlab ode solver with 4th-order Runge-Kutta method. To the study the performance of the controller developed in section IV, we choose a moderately aggressive trajectory defined in flat outputs as,

\[
x_n(t) = \begin{bmatrix} a_x(1 - \cos(2f_1 \pi t)) \\
a_y \sin(2f_2 \pi t) \\
a_z \cos(2f_3 \pi t) \end{bmatrix}, \quad \psi(t) = 0,
\]

where \(a_x = 2, a_y = 2.5, a_z = 1.5, f_1 = \frac{1}{4}, f_2 = \frac{1}{5}, f_3 = \frac{1}{7}\).

The rest of the states and the nominal feedforward inputs required to track the trajectory are calculated through differential flatness. These states refer to the desired states used in calculating the errors and the values of \(A, B\) in \((35)\). The controller performance is tested through several simulation tests. In the simulation environment, the mass parameters of the quadrotor are given by \(m_Q = 0.85kg, J = \)
For the LQR controller, the weight matrices $Q$ and $R$ are chosen as,

$$Q_1 = \text{diag}([Q_{11}, Q_{12}, Q_{13}, Q_{14}]),$$

where $Q_{11} = 0.5I_6$, $Q_{12} = 0.75I_6$, $Q_{13} = I_{3n}$, $Q_{14} = 0.75I_{3n}$ and $Q_2 = 0.2I_4$, $T = 0.01 \cdot I_{12+6n}$.

<table>
<thead>
<tr>
<th>TABLE III: Initial conditions for different trials</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
</tr>
<tr>
<td>$x_{i0}$</td>
</tr>
<tr>
<td>$v_{i0}$</td>
</tr>
<tr>
<td>$\omega_{i0}$</td>
</tr>
</tbody>
</table>

Fig. 5 draws out the trajectories of the controlled system for three different initial conditions. The initial conditions are listed in Table III. From Fig. 5, the trajectories for all three initial conditions converges to the reference trajectory, even for the initial condition with large initial deviation. This implies that the controller is still able to stabilize the trajectory to the reference, emphasizing that the linear controller developed through variation on manifolds has a large domain of attraction.

As it can be seen from the figure, the position tracking error $\delta x_i$ for load, Fig. 6a, rotation error $\Psi_R$ for quadrotor orientation, Fig. 6b and the orientation error $\Psi_i$ for $5^{th}$ link, Fig. 6c converge to zero. This validates that the controller developed for trajectory tracking of a flexible cable, through variation based linearization.

VI. CONCLUSIONS

We study the payload transportation problem of multiple quadrotors with the payload suspended through flexible cables. In particular we have considered the following systems: (a) single quadrotor with a point-mass payload suspended through a flexible cable; (b) single quadrotor with tethered to the ground through a flexible cable; (c) multiple quadrotors with a shared point-mass payload suspended through flexible cables; and (d) multiple quadrotors with a shared rigid-body payload suspended through flexible cables. For each of these systems, we have developed the Newton-Euler coordinate-free dynamic models and proven that the resulting dynamics are differentially-flat. For the single quadrotor with a point-mass payload suspended through a flexible cable with five links (16 degrees-of-freedom and 12 degrees-of-underactuation), we have used the coordinate-free dynamics to develop a geometric variation-based linearized equations of motion about a desired trajectory. We show that a finite horizon linear quadratic regulator, designed based on the linearized dynamics, can be used to track a desired trajectory with a relatively large region of attraction. We demonstrate this through several numerical simulations.

REFERENCES


[17] K. Sreenath and V. Kumar, “Dynamics, control and planning for cooperative manipulation of payloads suspended by cables from multiple quadrotor robots,” in Robotics: Science and Systems (RSS), 2013, this paper won the RSS Best Paper Award 2013.


