Enhancing Feasibility and Safety of Nonlinear Model Predictive Control with Discrete-Time Control Barrier Functions

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Abstract—Safety is one of the fundamental problems in robotics. Recently, one-step or multi-step optimal control problems for discrete-time nonlinear dynamical system are formulated to offer tracking stability using control Lyapunov functions (CLFs) while subject to input constraints as well as safety-critical constraints using control barrier functions (CBFs). The limitations of these existing approaches are mainly about feasibility and safety. In the existing approaches, the optimization feasibility and the system safety cannot be enhanced at the same time theoretically. In this paper, we propose two formulations that unifies CLFs and CBFs under the framework of nonlinear model predictive control (NMPC). In the proposed formulations, safety criteria is commonly formulated as CBF constraints and stability performance is ensured with either a terminal cost function or CLF constraints. Relaxing variables are introduced on the CBF constraints to resolve the tradeoff between feasibility and safety so that they can be enhanced at the same. The advantages about feasibility and safety of proposed formulations compared with existing methods are analyzed theoretically and validated with numerical results.

B. Related Work

Designing controllers to ensure provable safety guarantees for autonomous systems is vital. One approach to provide safety guarantees in control problem is to draw inspirations from control barrier functions [5]. The CBF-QP formulation [6] permits us to find the minimum perturbation for a given feedback controller to guarantee safety. Control Lyapunov functions (CLFs) [7] can be applied to stabilize the closed-loop dynamics of both linear and nonlinear dynamical systems [8]. Together with CBFs, the CLF-CBF-QP formulation [2] enables handling safety-critical constraints effectively in real-time. This approach is also generalized for high-order systems [9], [10]. Robust or adaptive optimal control are also applied with this technique [11]–[14].

1) Optimal Control with Discrete-time CBFs: Besides the continuous-time domain, the formulations of CBFs are generalized into discrete-time systems. An optimization problem can be formulated to calculate the current optimal control input, proposed in the DCLF-DCBF formulation in [3]. A type of model predictive control is also recently introduced to enhance performance. The model predictive control with control Lyapunov functions (CLF-NMPC) is proposed to ensure stability in [15], where CLF constraints are considered under nonlinear model predictive control (NMPC). A control design (MPC-CBF) for safety-critical tasks is firstly presented in [4], where the safety-critical constraints are enforced by discrete-time control barrier functions. This approach could also be applied to a multi-layer control framework [16], [17], where the safety-critical control with discrete-time CBF serves as a mid-level controller or planner.

2) Feasibility & Safety: Among the formulations in discrete-time domain, the optimal control with discrete-time CBFs also encounter feasibility issues. The infeasibility issues come from the potential empty intersection between the reachable set and the safe region confined by CBF constraints at each time step. Moreover, there exists a tradeoff between safety performance and feasibility and they cannot be enhanced at the same time, discussed in [4]. In other words, reducing decay rates of CBF constraints increases the system safety, but comes at the possibility of infeasibility. A potential way to partly handle the infeasibility issue is to adopt the CBF constraint only on the one-step, presented in the formulation MPC-GCBF in [18]. This approach is shown to enhance the feasibility, however, we are still under the restriction between choosing feasibility and safety, as the intersection between the reachable set and the safe region at the first step could still potentially be empty.

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I. INTRODUCTION

A. Motivation

Safety-critical optimal control and planning is one of the fundamental problems in robotic applications. In order to ensure the safety of robotic systems while achieving optimal performance, the tight coupling between potentially conflicting control objectives and safety criteria is considered in an optimization problem. Some researchers formulate this problem using control barrier functions under the continuous dynamics of the system [1], [2], where the optimal performance is achieved by the control Lyapunov functions and safety criteria is guaranteed through control barrier functions. Recently, this methodology is also introduced in the discrete-time domain and the optimal control problem can be formulated to calculate the current optimal input [3] or a sequence of ones in the fashion of model predictive control [4]. However, feasibility and safety are still the main challenges for these formulations using discrete-time control barrier functions. In this paper, our proposed formulations focuses on how to handle the these problems and provides a detailed comparison analysis compared to the existing methods using discrete-time control barrier functions.
Similarly, feasibility is challenging to be guaranteed in the continuous domain. Many existing approaches relax the CLF constraint to resolve the conflict between CLF and CBF constraints, summarized in [5]. However, the QP-based problem could become infeasible as the CBF constraint might violate the input constraint even when we relax the CLF constraint. In [2], a valid CBF is specifically designed for the adaptive cruise control scenario based on the system dynamics under input constraints, which could ensure the feasibility in the optimization. This approach solves the problem specifically for this system but not for general nonlinear systems. Recently in [19], the decay rates of CBF constraints, summarized in [5]. However, the QP-based problem could become infeasible as the CBF constraint might violate the input constraint even when we relax the CLF constraint. In [2], a valid CBF is specifically designed for the adaptive cruise control scenario based on the system dynamics under input constraints, which could ensure the feasibility in the optimization. This approach solves the problem specifically for this system but not for general nonlinear systems. Recently in [19], the decay rates of CBF constraints, summarized in [5]. However, the QP-based problem could become infeasible as the CBF constraint might violate the input constraint even when we relax the CLF constraint. In [2], a valid CBF is specifically designed for the adaptive cruise control scenario based on the system dynamics under input constraints, which could ensure the feasibility in the optimization. This approach solves the problem specifically for this system but not for general nonlinear systems. Recently in [19],

We propose two control frameworks for guaranteeing stability and safety using nonlinear MPC (NMPC), where the safety criteria are considered as discrete-time CBF constraints, and stability criteria appear as CLFs formulated either as a terminal cost or discrete-time constraints.

The decay rates of the discrete-time control barrier function constraints are relaxed in the optimization problem, which allow the proposed formulations to enhance the feasibility and the safety at the same time.

The proposed formulations are shown theoretically to enhance the feasibility and safety performance compared with existing approaches, and also validated with numerical examples.

D. Paper Structure

The paper is organized as follows. A brief background about CLFs and CBFs is presented and the existing optimal control formulations are revisited in Sec. [II]. The proposed formulations are illustrated in Sec. [III] which unifies CLF and CBF using NMPC. The advantages about feasibility and safety compared with the state-of-the-art are illustrated theoretically in Sec. [V] Numerical simulations are shown in Sec. [VI] to validate our approach. Concluding remarks are provided in Sec. [VII].

II. BACKGROUND

A. CLFs and CBFs

In this paper, we consider a discrete-time control system as follows,

\[
x_{t+1} = f(x_t, u_t),
\]

with control input \( u \) confined by admissible input set \( \mathcal{U} \). For safety-critical control, we consider a set \( \mathcal{C} \) defined as the superlevel set of a continuously differentiable function \( h: \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R} \),

\[
\begin{align*}
C &= \{ x \in \mathbb{R}^n : h(x) \geq 0 \}, \\
\partial C &= \{ x \in \mathbb{R}^n : h(x) = 0 \}, \\
\text{Int}(\mathcal{C}) &= \{ x \in \mathbb{R}^n : h(x) > 0 \}.
\end{align*}
\]

Throughout this paper, we refer to \( C \) as a safe set. The function \( h \) becomes a control barrier function in the discrete-time domain if it satisfies the relation as follows,

\[
\Delta h(x_k, u_k) = -\gamma_k h(x_k), \quad 0 < \gamma_k \leq 1,
\]

where \( \Delta h(x_k, u_k) := h(x_{k+1}) - h(x_k) \). Satisfying constraint \( \{3\} \), we have \( h(x_{k+1}) \geq (1 - \gamma_k)h(x_k) \), i.e., the lower bound of control barrier function \( h(x) \) decreases exponentially at time \( k \) with the rate \( 1 - \gamma_k \).

Besides the system safety, we are also interested in stabilizing the system with a feedback control law \( u \) under a control Lyapunov function \( V \) in the discrete-time domain,

\[
\Delta V(x_k, u_k) = -\alpha_k V(x_k), \quad 0 < \alpha_k \leq 1,
\]

where \( \Delta V(x_k, u_k) := V(x_{k+1}) - V(x_k) \). Similarly as above, the upper bound of control Lyapunov function decreases exponentially at time \( k \) with the rate \( 1 - \alpha_k \).

B. Existing Approaches Revisited

In this section, we will revisit some existing optimal control formulations using discrete-time CLFs or CBFs.

1) DCLF-DCBF [3]: The discrete-time control Lyapunov function and control barrier function can be unified into one optimization program, which achieves the control objective and guarantees system safety. This formulation was introduced in [3] and is presented as follows,

\[
\begin{align*}
\mathbf{u}_k^* &= \arg\min_{(u_k, s) \in \mathbb{R}^{m+1}} \mathbf{u}_k^T H(x)\mathbf{u}_k + \phi(s) \quad (5a) \\
\Delta V(x_k, u_k) + \alpha_k V(x_k) &\leq s \quad (5b) \\
\Delta h(x_k, u_k) + \gamma_k h(x_k) &\geq 0 \quad (5c) \\
u_k &\in \mathcal{U}, \quad (5d)
\end{align*}
\]

where \( H(x) \) is any positive definite matrix and \( s \geq 0 \) is a slack variable together with additional cost term \( \phi(s) \geq 0 \) that allows the Lyapunov function to grow when the CLF and CBF constraints are conflicting. The safe set \( \mathcal{C} \) in \( \{2\} \) is invariant along the trajectories of the discrete-time system with controller \( \{3\} \) if \( h(x_0) \geq 0 \) and \( 0 < \gamma_k \leq 1 \).

2) MPC-CBF [4]: Inspired by the previous work of model predictive control and control barrier functions, DCLF-DCBF can be improved by taking future state prediction into account, yielding the form of MPC-CBF firstly introduced in [4] and shown as follows,
MPC-CBF:

where $U = [u_{t}^T, ..., u_{t+N-1}^T]^T$. In the cost function, $p(x_{t+N}|t)$ and $q(x_{t+k}|t, u_{t+k}|t)$ represent terminal cost and stage cost at each time step. For constraints, (6b) describes the system dynamics, (6c) shows the input constraints along the horizon and (6d) represents the initial condition constraint. The CBF constraints imposed in (6e) are designed to guarantee the forward invariance of the safe set $C$ associated with the discrete-time control barrier function. Here we have

$$\Delta h(x_{t+k}|t, u_{t+k}|t) = h(x_{t+k+1}|t) - h(x_{t+k}|t).$$

The optimal solution to (6) at time $t$ is a sequence of inputs as $U^* = [u_t^T, ..., u_{t+N-1}^T]^T$. Then, the first element of the optimizer vector is applied,

$$u(t) = u_{t|t}^*(x_t).$$ (7)

This constrained finite-time optimal control problem (6) is repeated at time step $t+1$, based on the new state $x_{t+1|t+1}$, yielding a receding horizon control strategy.

3) MPC-GCBF [18]: In the formulation (6), the CBF constraints are imposed on multi-steps along the horizon, which increases the burden of complexity with a large horizon and the possibility of infeasibility also increases with more constraints. One way to suppress the complexity and enhance the feasibility is to apply an one-step CBF constraint, which is done by modifying the CBF constraint in (6e) as follows,

$$\Delta h(x_{t+1}|t, u_{t|t}) \geq -\gamma h(x_{t|t}).$$ (8)

This approach is firstly shown in the MPC-GCBF formulation in [18], and enhances the feasibility and reduces the computational time at the same time due to fewer constraints. This approach is also generalized with consideration over the high relative-degree constraint, where constraints in (6e) are modified as constraints posed on two nonadjacent steps,

$$h(x_{t+m}|t) \geq (1 - \gamma)^m h(x_{t|t}),$$ (9)

where $m$ represents the relative degree of the high-order safety constraint.

4) CLF-NMPC [15]: Besides the control barrier functions, the model predictive control is also unified with control Lyapunov functions, where stability constraints with CLFs are considered in the proposed CLF-NMPC formulation [15]. The CLF constraint can be imposed during one-step or multi-step,

$$\Delta V(x_{t+k}|t) \leq -\alpha_k V(x_{t+k}|t) + s_k,$$ (10)

where $s_k$ is the slack variable. The safety criteria is not considered in CLF-NMPC. For more details see [15].

III. CLFS AND CBFs UNIFIED WITH NMPC

In this section, we are going to present two proposed formulations: CBF-NMPC and CLF-CBF-NMPC unifying CLFs and CBFs with NMPC, which will be shown in Sec. II-A

A. Formulations

Firstly, we incorporate CBF constraints of relaxing decay rates into a nonlinear model predictive control framework, denoted as CBF-NMPC and shown as follows,

CBF-NMPC:

$$J_t^U(x_t) = \min_{U_{t:k}} \beta V(x_{t+N}|t) + \sum_{k=0}^{N-1} q(x_{t+k}|t, u_{t+k}|t) + \psi(\omega_k)$$ (11a)

s.t. $x_{t+k+1} = f(x_{t+k}|t, u_{t+k}|t), k = 0, ..., N-1$ (11b)

$u_{t+k}|t \in \mathcal{U}, x_{t+k}|t \in \mathcal{X}, k = 0, ..., N-1$ (11c)

$x_{t|t} = x_0$ (11d)

$h(x_{t+k+1}|t) \geq \omega_k(1 - \gamma_k)h(x_{t+k}|t), \omega_k \geq 0$ (11e)

for $k = 0, ..., M_{\text{CBF}}-1$

where the decision variables are $U = [u_t^T, ..., u_{t+N-1}^T]^T$ and $\Omega = [\omega_1^T, ..., \omega_{M_{\text{CBF}}-1}]^T$. The system dynamics (11b), the input constraint (11c) and the initial condition (11d) are imposed on the optimization. The control Lyapunov function $V(x_{t+N}|t)$ is used as a terminal cost scaled up with the parameter $\beta$, together with the cumulative stage cost along the horizon $\sum_{k=0}^{N-1} q(x_{t+k}|t, u_{t+k}|t)$. The terminal cost as CLF adopts the fashion of work from the field of MPC, as noted in [20], where stability usually can be achieved without the need to specify a terminal state constraint if $\beta$ is selected large enough.

Different from the formulation in (9), the decay rates of control barrier functions are relaxed from the fixed value $1 - \gamma_k$ into optimal variables $\omega_k(1 - \gamma_k)$ and an additional cost about the relaxing rate variables $\psi(\omega_k) \geq 0$ is included in the optimization. This function $\psi$ can be tuned for different performance. The relaxing rate variable $\omega_k$ is constrained by (11e) such that the following relation is guaranteed,

$$h(x_{t+k+1}|t) \geq \omega_k(1 - \gamma_k)h(x_{t+k}|t) \geq 0.$$ (12)

This results in the safety guarantee for the first $M_{\text{CBF}}$ steps in the open-loop trajectory but not for the entire horizon $N$. Here, one horizon length $N$ is designed for dynamics constraint, input constraint and stage cost, and another horizon length $M_{\text{CBF}} \leq N$ is applied for CBF constraints, which allows us to choose the appropriate value of $M_{\text{CBF}}$ to reduce the computational complexity. We denote this formulation as CBF-NMPC as the optimization will become nonlinear with either nonlinear dynamics or nonlinear CBF.
**Remark 1.** Here, we hypothesize that the closed-loop trajectory can still be guaranteed by iterations. Formal guarantee of this property requires analysis of recursive feasibility and reachability, which will be proved in the subsequent work.

**Remark 2.** CBF-NMPC represents a generalized form of MPC-CBF and MPC-GCBF with the technique of relaxing decay rates of safety constraints. CBF-NMPC becomes similar to MPC-CBF when $M_{\text{CBF}} = N$ and similar to MPC-GCBF when $M_{\text{CBF}} = 1$. When the relaxing variable $\omega_k$ is fixed as 1, CBF-NMPC becomes the same as MPC-CBF when $M_{\text{CBF}} = N$ and the same as MPC-GCBF when $M_{\text{CBF}} = 1$.

**Remark 3.** The fixed decay rates for safety constraints existing in MPC-CBF and MPC-GCBF are relaxed and become as optimal variables in CBF-NMPC, which increase the optimization feasibility.

**Remark 4.** MPC-GCBF reduces the computational complexity and increases feasibility by reducing multi-step constraints into one-step. However, one-step constraint might not confine the system sufficiently and the optimization problem may become infeasible after a while in the closed-trajectory, shown in Fig. [4b]. Moreover, its set invariance for high-relative degree system, we just simply need $M_{\text{CBF}} > m$, where $m$ represents the relative-degree defined in [9].

Alternatively, the stability criteria with CLF could be posed as constraints instead of as a terminal cost, which leads to the formulation as follows,

**IV. THEORETICAL ANALYSIS**

In this section, we are going to illustrate theoretically the advantages about feasibility and safety of the proposed approaches. These advantages compared with the state-of-the-art are summarized in Table I.

**A. Theoretical Analysis of Feasibility**

In this section, we are going to illustrate the enhancement of feasibility with reachability analysis by comparing MPC-CBF and CBF-NMPC.

For the MPC-CBF formulation, the reachable set and safe region confined by the CBF constraint at each time are defined as follows. The optimization of MPC-CBF is feasible when the intersections between the reachable set $\mathcal{R}_{k}^{\text{MPC-CBF}}$ and safe set $\mathcal{S}_{k}^{\text{MPC-CBF}}$ at time $t + k$ are non-empty for all $k$, where

$$\mathcal{R}_{k}^{\text{MPC-CBF}} = \{ x_{t+k} \in \mathbb{R}^n : \exists i = 0, ..., k, x_{t+i} = f(x_{t+i}, u_{t+i}), u_{t+i} \in \mathcal{U}, x_{t+k} \in \mathcal{X}, x_{t} = x_{t} \},$$

$$\mathcal{S}_{k}^{\text{MPC-CBF}} = \{ x_{t+k} \in \mathbb{R}^n : h(x_{t+k}) \geq \inf_{x \in \mathcal{R}_{k}^{\text{MPC-CBF}}} h(x) \},$$

(14)

We denote the safe region at each time step as $\mathcal{S}_{k}^{\text{MPC-CBF}}$, but notice that it also depends on the value of optimal value $x_{t+k-1}$, which depends on the states and the inputs of previous nodes before the index $k-1$. $\mathcal{S}_{k}^{\text{MPC-CBF}}$ could be rewritten as a function of reachable set at the one-step before,

$$\mathcal{S}_{k}^{\text{MPC-CBF}} = \{ x_{t+k} \in \mathbb{R}^n : h(x_{t+k}) \geq (1-\gamma_k) \inf_{x \in \mathcal{R}_{k-1}^{\text{MPC-CBF}}} h(x) \},$$

(15)

as [15] leads to the following equation being valid

$$h(x_{t+k}) \geq (1-\gamma_k) h(x_{t+k-1}),$$

with at least one value of $x_{t+k-1}$.

For the CBF-NMPC formulation, the reachable set is the same as MPC-CBF as they share the same initial condition, system dynamics and input constraints, i.e.,

$$\mathcal{R}_{k}^{\text{CBF-NMPC}} = \mathcal{R}_{k}^{\text{MPC-CBF}}.$$
The system safety can be influenced by many factors, including the safety function, the cost function, and other hyperparameters, etc. In this section, we focus on the influence from the hyperparameter $\gamma$ and the additional cost function $\psi(\omega_k)$ for the decay-rate relaxing variable $\omega_k$.

By reducing $\gamma_k$ for MPC-CBF / MPC-GCBF / DCLF-DCBF, the system safety will increase as the smaller $\gamma_k$ represents a slower decay rate of lower bound of control barrier function, see [3]. However, from (16), we can see that reducing $\gamma_k$ for MPC-CBF makes $S_{\text{MC-CBF}}^{\text{MPCCBF}}$ smaller, which leads to the optimization more likely to be infeasible along the trajectory as the intersection between the reachable set and the region constrained by safety constraint decreases. This leads to a tradeoff between feasibility and safety. This tradeoff also happens among DCLF-DCBF and MPC-GCBF with similar reasons and forces us to choose either feasibility or safety for performance. However, in the case of our proposed approach, the region confined by safety constraint won’t be affected by changing the value of $\gamma_k$, shown in (18). Hence, the intersection between the reachable set and the region constrained by safety constraint is independent of $\gamma$. This allows us to enhance the safety by reducing $\gamma_k$ while not harming feasibility, which resolves the tradeoff between feasibility and safety.

The design of the additional cost function $\psi(\omega_k)$ for the decay-rate relaxing variable $\omega_k$ could also affect the safety performance. For example, the function $\psi(\omega_k)$ be in the form as follows,

$$\psi(\omega_k) = P_\omega(\omega_k - 1)^2$$

which keeps $\omega_k$ close to 1 and thus minimizes the deviation of the CBF constraint from the nominal decay rate of $1 - \gamma_k$. When the hyperparameter $P_\omega$ becomes larger, the optimized value of $\omega$ tends to be closer to 1, which implies the deviation from the nominal decay rate $1 - \gamma_k$ is smaller. Numerically, $\gamma_k$ tends to be optimized as value smaller than 1 to increase the safe region [13] confined by CBF constraint at each time step. When $\omega_k = 0$, the relaxed CBF constraint becomes equivalent to a simple distance constraint and MPC with distance constraints (MPC-DC [4]) needs longer horizon to generate an expected obstacle avoidance performance in a closed-loop trajectory. Therefore, it’s not recommended to set a relatively too small value for $P_\omega$, which would over-relax the CBF constraint, i.e., the optimized value of $\omega_k$ could be too small.

To sum up, by reducing $\gamma_k$ and utilizing an appropriate form of the additional cost function for decay-rate relaxing variable $\omega_k$, the proposed approach would outperform the existing formulations in term of safety while not harming feasibility performance.

### V. Numerical Examples & Results

In this section, we going to show numerical results to illustrate the advantages of our proposed formulations with respect to the existing approaches. Consider the discrete-time linear triple-integrator system,

$$x_{k+1} = Ax_k + Bu_k$$

where $x = [x, v, a]^T$ and $u = [j]^T$ represent position ($x$), velocity ($v$), acceleration ($a$), and jerk ($j$), respectively. The admissible input set is $U = \{j \in \mathbb{R} : j_{\min} \leq j \leq j_{\max}\}$.

For numerical simulations in this section, we set the sampling time as $\Delta t = 0.1s$ together with input lower and upper bounds as $j_{\min, \max} = -1m/s^3, 1m/s^3$. All simulations run in MATLAB and the optimal control is formulated with Yalmip [21] as modelling language and solved with IPOPT [22].
Fig. 1: Feasibility comparison with \( h(x) = -x \) between MPC-CBF \((N = 8)\) and CBF-NMPC \((N = 8, M_{\text{CBF}} = 8)\) with different values of \( \gamma \). Feasible states are marked by red points (MPC-CBF) and blue circles (CBF-NMPC). It’s shown that the feasibility region of MPC-CBF is always a subset of feasibility region of CBF-NMPC, and the feasibility region of CBF-NMPC is independent of \( \gamma \).

Fig. 2: Feasibility comparison with \( h(x) = -x \) between MPC-GCBF \((N = 8)\) and CBF-NMPC \((N = 8, M_{\text{CBF}} = 3)\) with different values of \( \gamma \). It’s shown that the feasibility region of MPC-GCBF is always a subset of feasibility region of CBF-NMPC, and the feasibility region of CBF-NMPC is independent of \( \gamma \).

Fig. 3: Feasibility comparison with \( h(x) = x^2 + v^2 + a^2 - 1 \) between DCLF-DCBF and CLF-CBF-NMPC \((N = 8, M_{\text{CBF}} = 8, M_{\text{CLF}} = 8)\) with different values of \( \gamma \). The zero-level set of CBF constraint is marked in yellow. It’s shown that the feasibility region of DCLF-DCBF is always a subset of feasibility region of CLF-CBF-NMPC, and the feasibility region of CLF-CBF-NMPC is independent of \( \gamma \).

A. Numerical Results for Feasibility

Our proposed formulations along with existing ones is compared by solving the optimization problems at all sampling states in a closed space. Precisely, we iterate over sampling states in the closed space \( \mathcal{X} \) as

\[
\mathcal{X} = \{(x, v, a) \in \mathbb{R}^3 : x_{\min} \leq x \leq x_{\max}, v_{\min} \leq v \leq v_{\max}, a_{\min} \leq a \leq a_{\max}\}
\]

and run these optimal controllers to see whether the optimization problems are feasible at a given state \( x_k \). For simulation, we set \( x_{\min,\max} = -2m, 0m, v_{\min,\max} = 0m/s, 2m/s \) and \( a_{\min,\max} = 0m/s^2, 2m/s^2 \). All the feasibility performance comparison is evaluated between approaches with the same horizon \( N \) and same form of stage cost and terminal cost.

To compare the feasibility performance among MPC-CBF, MPC-GCBF and CBF-NMPC, we choose a high-order control barrier function

\[
h(x) = -x,
\]

which enforces the system to stay on one side of the yz plane \((x \leq 0)\). In Figs. 1-2 results from CBF-NMPC are compared with MPC-CBF and MPC-GCBF for feasibility analysis. The comparisons are validated with different values of \( \gamma = 0.05, 0.10, 0.15, 0.20 \). For a reasonable comparison, the horizon length of CBF constraints is assumed as \( M_{\text{CBF}} = 3 \) for CBF-NMPC to compare with MPC-GCBF, as the relative-degree of the CBF in Eq. 7 is 3 for a triple integrator system. The formulations of MPC-CBF and MPC-GCBF are shown to enhance the feasibility with larger value of \( \gamma \).
CBF-NMPC does enhance the feasibility compared with MPC-CBF as more states are feasible for MPC-GCBF with any value of $\gamma_k$. The feasibility of the proposed CBF-NMPC is shown to consistently outperform MPC-CBF and MPC-GCBF for any value of $\gamma_k$, where the feasible state region for MPC-CBF or MPC-GCBF lies always inside the one for CBF-NMPC. Additionally, the feasible state region of CBF-NMPC is independent of the value of $\gamma_k$, shown in Figs. [1] [2] which verifies $S_{\text{cbf},k}^{\text{CBF-NMPC}}$ is independent of $\gamma_k$ as shown in [18].

To compare the feasibility performance between DCLF-DCBF and CLF-CBF-NMPC, we choose a relative-degree one control barrier function

$$h(x) = x^2 + v^2 + a^2 + 1,$$

as the DCLF-DCBF method can only optimize one-step control input. The comparison result is shown in Fig. [3] where CLF-CBF-NMPC outperforms DCLF-DCBF in terms of feasibility for any values of $\gamma_k$. Similar to what we have seen previously, DCLF-DCBF enhances the feasibility with larger $\gamma_k$, while the feasible state region for CLF-CBF-NMPC is independent of $\gamma_k$. Notice that the unsafe states, which are inside the sphere region defined by (24), are excluded from the state sampling test for feasibility. The zero-level surface of the control barrier function is colored in yellow in Fig. [3].

We also remark that the numbers of safety constraints for CBF-NMPC and CLF-CBF-NMPC are larger than the ones for MPC-GCBF and DCLF-DCBF, but the feasibility performance are enhanced in our proposed approach, which demonstrates the importance of decay-rate relaxing technique that are introduced in the two proposed formulations. To sum up, the proposed formulations outperform the state-of-the-art in terms of feasibility.

B. Safety

The safety performance between controllers are compared numerically in this section. Given the same initial condition $x_0 = [-2.0, 0.0, 1.0]^T$, we test each controller performance by using hyperparameters $\gamma_k = 0.05, 0.10, 0.15, 0.20$. The results for comparison among these approaches are shown in Fig. [4]. Among MPC-CBF and CBF-NMPC, it can be seen that by reducing the value of $\gamma_k$, the value of CBF decreases slower, which implies a safer closed-loop trajectory. We notice that CBF-NMPC is the only approach that the feasibility is held along the trajectory with $\gamma = 0.05$, while other two already become infeasible at the initial condition. Additionally, MPC-GCBF has shown to have disadvantage over feasibility in a close-loop trajectory, illustrated in Fig. [4b].

We can see that MPC-GCBF becomes infeasible after around 5 seconds in a trajectory starting from the initial condition. This comes from the one-step constraint doesn’t sufficiently confine the system for safety and leads the system into an infeasible state after a while. We also notice that the control barrier function for CBF-NMPC with $\gamma_k = 0.20$ is larger than $\gamma_k = 0.15$ after $t = 8s$, this happens because due to numerical errors. This comes from that, after $t = 8s$, the CBF $h(x)$ is almost zero and its derivative becomes almost zero and the solver tends to optimize the additional cost for relaxing decay-rate variable instead of the stage and terminal cost. Together with feasibility analysis in Sec. V, we have shown that by reducing $\gamma_k$, CBF-NMPC could enhance the safety of the closed-loop trajectory while not adversely affecting feasibility, which resolves the trade-off between feasibility and safety.

VI. CONCLUSION & FUTURE WORK

In this paper, we propose formulations to unify control Lyapunov function and control barrier functions under the framework of nonlinear model predictive control. Compared with previous work, the decay-rate of the CBF constraints are relaxed and different horizon lengths are considered for the cost function and the constraints. The proposed formulations are shown both theoretically and numerically to outperform the state-of-the-art from the perspective of feasibility and safety.

Our future work will focus on how to implement the proposed formulations either as a mid-level planner or a real-time controller on the mobile robots, where modelling uncertainty, system disturbance and noise are required to be considered for real-time deployment. From the theoretical perspective, a formal discussion about recursive feasibility...
and stability will be carried out to summarize the technique of nonlinear MPC with control barrier function.

REFERENCES


