Abstract—This paper works towards merging two popular approaches in the control community for safety control: Hamilton-Jacobi (HJ) reachability analysis and Control Barrier Functions (CBFs). HJ reachability analysis provides a direct construction of value functions that provide safety guarantees and safe controllers, however the online implementation can be overly conservative and/or rely on jumpy bang-bang control. The CBF community has methods for safe-guarding controllers in the form of point-wise optimization programs such as CBF-QP, where the CBF-based safety certificate is used as a constraint. However, finding a valid CBF for a general dynamical system is challenging. This paper merges these two methods by introducing a new reachability formulation inspired by the structure of CBFs that constructs a Control Barrier-Value Function (CBVF). We verify that CBVF is a viscosity solution to a novel Hamilton-Jacobi-Isaacs Variational Inequality and preserves the same safety guarantee as the original reachability formulation. Finally, we propose an online Quadratic Program formulation whose solution is always an optimal control signal, which is inspired by the CBF-QP. We demonstrate the benefit of using the CBVFs for double-integrator and Dubins car systems by comparing it to previous methods.

I. INTRODUCTION

A. Motivation & Related Work

Value function-based approaches are common techniques for solving safe control problems. Two such methods are Hamilton-Jacobi (HJ) reachability analysis and Control Barrier Functions (CBFs). HJ reachability analysis formulates the reachability of a target set as an optimal control problem, and has long been used as a formal theoretical tool for safety analysis and synthesis of safe controllers [1], [2]. HJ reachability-based value functions can be solved numerically by using the dynamic programming principle [3]. The zero-superlevel set of the value function describes the safe set, and the optimal safety controller can be synthesized based on the gradient of the function. Moreover, the safe control can be robust to disturbances [2].

The main drawbacks of HJ reachability analysis are twofold. First, although there have been recent advances to improve computational efficiency [4], [5], [6], most numerical methods to construct the value function suffer from the curse of dimensionality [7]. Secondly, the resulting safe optimal control policy is generally overly conservative when applied directly. A popular remedy for reducing conservativeness is to use a least-restrictive hybrid controller—the optimal control is only applied when the system is very close to the

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in the maximal safe region for a desired safety constraint, whereas CBFs often can only provide a conservative estimate of safe region. On the other hand, online CBF-based safety controllers like CBF-QP are a powerful tool to apply CBFs to high-dimensions systems in real-time applications, whereas the HJ reachability suffers from the curse of dimensionality and its value function’s online deployment is not straightforward due to the optimal policy’s restrictive behavior. A recent paper uses HJ reachability functions as CBFs [13], but a theoretical understanding of the relationship between the two methods is still lacking.

B. Paper Organization and Contributions

In light of the fact that reachability-based value functions and CBFs are tackling a similar problem in complementary ways, we merge the two functions in a theoretical way. First, in Sec. II we briefly summarize and compare the concept of a value function from HJ reachability and CBFs.

In Sec. III we introduce the notion of a Robust Control Barrier-Value function (CBVF) that merges reachability-based value functions and CBFs into one function. This function (a) can be used for finite-time safety guarantees, (b) is robust to bounded disturbances, (c) recovers the maximal safe set for a desired safety constraint, and (d) leads to a safety control that satisfies the control bound everywhere inside the safe set. The main theoretical contribution is a verification that the CBVF is a viscosity solution of a particular Hamilton-Jacobi-Isaacs variational inequality (HJI-VI), and this can be used to numerically construct a valid CBVF. This constructive method does not naturally scale well, but can benefit from methods from the reachability community that enhance scalability [4], [5], [17].

In Sec. IV, we introduce the optimal control policy corresponding to the CBVF which is less conservative than that from the original HJ reachability, and is less jerky than using least-restrictive control. For systems affine in control and disturbance, we show that such an optimal controller can be obtained by solving a quadratic program, namely the Robust CBVF-QP. In Sec. V, we demonstrate these findings by simulating a comparison between the CBVF and the original HJ reachability-based value function, and the resulting online trajectories and control signals. We also show that the new CBVF formulation is robust to bounded disturbances. Finally, the online trajectory from the Robust CBVF-QP for finite-time horizon navigation of a Dubins car is compared with previous HJ reachability and CBF based approaches. In Sec. VI, we provide concluding remarks.

II. BACKGROUND

A. Problem Formulation

Consider a state trajectory of the continuous-time time-invariant controlled system with disturbance, solving
\[ \dot{x}(s) = f(x(s), u(s), d(s)), s \in [t, t'], \quad x(t) = x, \]
where \( t \) and \( x \) are the initial time and state, respectively. \( u \in U \subset \mathbb{R}^n \) is the control input, \( d \in D \subset \mathbb{R}^w \) is the disturbance where \( U, D \) are compact and convex sets, and \( f : \mathbb{R}^n \times U \times D \rightarrow \mathbb{R}^n \) is Lipschitz continuous in the state and bounded. Let \( U_{[t, t']}, D_{[t, t']} \) be a set of Lebesgue measurable functions from the time interval \([t, t']\) to \( U \) and \( D \), respectively. For simplicity, we set the final time as 0. For every initial time \( t \leq 0 \), initial state \( x \in \mathbb{R}^n \), \( u(\cdot) \in U_{[t, 0]} \), and \( d(\cdot) \in D_{[t, 0]} \), system (1) admits a unique solution trajectory. We denote this trajectory as \( x(s) \), and will say that \( x(\cdot) \) solves (1) for \( (x, t, u, d) \) with a slight abuse of notation.

Throughout the paper, we assume that the disturbance signal \( d(\cdot) \) can be determined in reaction to the control signal in a form of a strategy \( \xi_d : U_{[t, 0]} \rightarrow D_{[t, 0]} \). However, we restrict it to draw only from nonanticipative strategies with respect to \( u(\cdot) \), denoted as \( \xi_d \in \Xi_{[t, 0]} \). Informally speaking, this means that until a current time \( s \), if \( u_1(\cdot) \) and \( u_2(\cdot) \) cannot be distinguished from each other, the resulting value of \( d(s) \) determined by \( \xi_d \in \Xi_{[t, s]} \) should be same [2].

Now consider a set \( \mathcal{L} \) defined as a zero-superlevel set of a bounded Lipschitz continuous function \( l : \mathbb{R}^n \rightarrow \mathbb{R} \):
\[ \mathcal{L} = \{ x : l(x) \geq 0 \}. \]  
(2)
The objective of the safety control is to guarantee the trajectory to stay in \( \mathcal{L} \) for \( s \in [t, 0] \) under the worst case disturbance. We refer to \( l(x) \) as the safety target function. More formally, we are interested in the following problems:

- **Computing the viability kernel** \( \mathcal{S}(t) \) for \( \mathcal{L} \): Verify \( \mathcal{S}(t) := \{ x \in \mathcal{L} : \forall \xi_d \in \Xi_{[t, 0]} \exists u(\cdot) \in U_{[t, 0]} \text{ s.t. } \forall s \in [t, 0] \}, x(s) \in \mathcal{L} \text{ where } x(s) \text{ solves } (1) \text{ for } (x, t, u, \xi_d) \} \) \( t < 0 \). \( \mathcal{S}(t) \) is the set of all the initial states at time \( t \) in \( \mathcal{L} \) from which there exists an admissible control signal that keeps the system safe under the worst-case disturbance. \( \mathcal{S}(t) \) is called the viability kernel [1] for \( \mathcal{L} \).

- **Computing a robust safe control** \( u(\cdot) \) for \( \mathcal{L} \): For each \( x \in \mathcal{S}(t) \), verify a control signal \( u(\cdot) \in U_{[t, 0]} \) that renders the trajectory safe for \( s \in [t, 0] \), under the worst-case disturbance.

B. Hamilton-Jacobi Reachability Analysis

It has been verified that solving for the viability kernel and the robust safe control signal can be posed as an optimal control problem, which can be solved using HJ reachability analysis [1], [2], [18]. First, we define a cost function as
\[ J(x, t, u(\cdot), d(\cdot)) := \min_{s \in [t, 0]} l(x(s)), \]  
(3)
which captures the minimal value of \( l(\cdot) \) along the trajectory \( x(\cdot) \) that solves (1) for \( (x, t, u, d) \). If \( \exists s \in [t, 0] \) such that \( J(x, t, u(s), d(s)) < 0 \), it means that the trajectory was violating the safety constraint at some point in the time horizon (obtaining a negative value of \( l \)), and is therefore unsafe. The objective of the safety control is to make \( J \) as big as possible, whereas under the worst case, the disturbance would act in a direction of decreasing \( J \) as much as it can. Based on this, we can define the value function \( V : \mathbb{R}^n \times (-\infty, 0] \rightarrow \mathbb{R} \) as
\[ V(x, t) := \min_{\xi_d \in \Xi_{[t, 0]} \, u \in U_{[t, 0]}} \max_{s \in [t, 0]} J(x, t, u(\cdot), \xi_d(\cdot)), \]  
(4)
Then, by the following proposition, the viability kernel for \( \mathcal{L} \) is \( \mathcal{S}(t) = \{ x \in \mathbb{R}^n : V(x, t) \geq 0 \} \). Note that the minimum and maximum in \( \Xi_{[t, 0]} \, U_{[t, 0]} \) always exists because \( U \) and \( D \) are compact and convex [19].
Proposition 1. For all $t \leq 0$, the viability kernel for $L, S(t)$, always is $\{x \in \mathbb{R}^n : V(x, t) \geq 0\}$.

Proof. This is directly from the definition of $V$ and $S(t)$. $\square$

Note that if $S(t)$ is empty, safety can never be guaranteed under the worst-case disturbance. In the complement of $S(t)$, the value function $V(x, t)$ is negative, therefore, for any admissible control, the trajectory is unsafe under the worst-case disturbance. This set $S(t)^\circ$ describes what is known in the HJI Reachability community as a Backward Reachable Tube of the unsafe set. Because of this, solving for $V(x, t)$ in (4) is regarded as a reachability problem [1].

The value function $V(x, t)$ is the viscosity solution to the following Hamilton-Jacobi-Isaacs Variational Inequality (HJI-VI) [18]:

$$0 = \min \left\{ l(x) - V(x, t), \quad \begin{align*}
D_t V(x, t) + \max_{u \in U} \min_{d \in D} D_x V(x, t) \cdot f(x, u, d) \end{align*} \right\}$$

(5)

with the terminal condition $V(x, 0) = l(x)$. This means that $V(x, t)$ can be computed directly by dynamic programming backwards in time by applying the HJI-VI at each point in the state space.

Remark 1. The viscosity solution $V(x, t)$ is a weak solution to (5): $V(x, t)$ is not differentiable for some $(x, t)$. Under the Lipschitz assumptions for the dynamics ($f$) and the cost ($l$) in the state, $V(x, t)$ is Lipschitz continuous, which is differentiable almost everywhere (a.e.) in $(x, t)$-space [20, Th.3.2.[21].

When the viability kernel $S(t)$ is non-empty, from any element in $S(t)$, we can synthesize a robust safe control signal from the optimal control policy. Based on whether the left or the right term in the minimum of (5) is active, the optimal policy $\pi^*_V(x, t) : \mathbb{R}^n \times (-\infty, 0] \to U$ is determined in a different way. That is, when $V(x, t) < l(x)$,

$$\pi^*_V(x, t) = \arg \max_{u \in U} \min_{d \in D} D_x V(x, t) \cdot f(x, u, d),$$

(6)

and the right term of (5) is 0. Second, when $V(x, t) = l(x)$, any element of $K_V(x, t) := \{u \in U : D_t V(x, t) + \min_{d \in D} D_x V(x, t) \cdot f(x, u, d) \geq 0\}$

(7)

can be used as $\pi^*_V(x, t)$. Therefore, the second case may allow multiple options for the optimal control. In either case, for any $d \in D$, $\dot{V}(x(t), t) = D_t V(x(t), t)$

$$+ D_x V(x(t), t) \cdot f(x(t), \pi^*_V(x(t), t), d) \geq 0,$$

where $x(\cdot)$ is an instantaneous trajectory of (1) at $t$ with $x(t) = x$, control $\pi^*_V(x, t)$ and disturbance $d$. Therefore, this implies that for any initial state $x \in S(t)$, for any $\xi_d \in \Xi_{[t, 0]}$, along the optimal trajectory $x^*(\cdot)$ which solves (1) for $(x, t, \pi^*_V, \xi_d[\pi^*_V])$, the value function $V(x^*(s), s)$ will never decrease. Since $V$ is non-negative at the initial time $t$, it is always kept non-negative under $\pi^*_V$ for $s \in [t, 0]$, which means the trajectory is rendered safe.

Remark 2. Note that for the second case of $\pi^*_V$, for any optimal $u \in K_V(x, t)$ and any $d \in D$,

$$\dot{V}(x(t), t) = \hat{V}(x(t), t) \geq 0,$$

(8)

where $x(\cdot)$ is an instantaneous trajectory of (1) at $t$ with $x(t) = x$, control $u$ and disturbance $d$. This means that for the second case, $\pi^*_V$ requires $l$ to increase, in other words, it never allows the trajectory to get closer to the safety boundary. Therefore, such optimal control policy is often too restrictive to be used as a safety filter for a reference control signal. In the reachability community, to remedy this, a common practice is to switch from the reference control to the safe optimal control only when $V(x(s), s)$ is close to 0, so called least-restrictive control law [17], [22], [23]. The resulting control system with such switching law may give undesirable jerky behaviors and is prone to errors in numerically computed $D_x V$.

C. Control Barrier Functions

An alternative approach for achieving the safety control objective is to use Control Barrier Functions (CBFs). The theory of CBFs is developed upon viability theory and Lyapunov-based stability theory [9].

Definition 1. Let $C$ be a zero-superlevel set of a continuously differentiable function $B : \mathbb{R}^n \to \mathbb{R}$. Consider a Lipschitz continuous controlled system without disturbance, $f = f(x(s), u(s))$. Then $B$ is a Control Barrier Function for this system if there exists an extended class $K_\infty$ function $\alpha$ such that for all $x \in C$,

$$\max_{u \in U} D_x B(x) \cdot f(x, u) \geq -\alpha(B(x)).$$

(9)

Introducing $-\alpha(B(x))$ on the right hand side of (9) is inspired by the condition that Control Lyapunov Functions (CLFs) should satisfy in order to provide exponential stabilizability [9]. In practice, a linear function $\gamma z (\gamma > 0)$ is often used as $\alpha(z)$. In this case, $\gamma$ serves as a maximal discount rate of $B(x(s))$. Informally, this means that $B(x(s))$ is not allowed to decay faster than the exponentially decaying curve $\dot{B}(x(s)) = -\gamma B(x(s))$, therefore potential unsafe behaviors smooth out as it approaches the safe boundary. More formally, the following holds:

Theorem 1. [9, Corollary 2] For such $B$ and its zero-superlevel set $C$, any Lipschitz continuous controller $\pi : C \to U$ such that $\pi(x) \in K_B(x)$ where

$$K_B(x) := \{u \in U : D_x B(x) \cdot f(x, u) \geq -\alpha(B(x))\},$$

(10)

will render the set $C$ forward invariant [9]. In other words, $C$ is control invariant.

Condition (9) can be incorporated in an online optimization based controller that minimizes the norm of the difference between $u$ and the reference control $u_{ref}$. For control-affine systems, this can become a Quadratic Program, namely Control Barrier Function-based Quadratic Program (CBF-QP) [9], and can be used as an online safety filter for any reference control signal $u_{ref}$. 
D. Comparison between HJ reachability and CBF

In this subsection, we restrict our interest to systems without disturbance, \( f = f(x(s), u(s)) \), for the comparison between value function from the reachability \( V \) and CBF \( B \). Note that by extending the definition of \( V \) to infinite-horizon as \( V_\infty(x) := \lim_{t \to -\infty} V(x, t) \), we can get a time-invariant value function [24] whose zero-superlevel set \( S_\infty := \{ x : V_\infty(x) \geq 0 \} \) is a maximal control invariant set contained in \( L \). The latter results from extending Proposition 1 to infinite horizon.

The geometric connection between the zero-superlevel set of the CBF \( B \), \( C \), and the zero-superlevel set of \( V_\infty \), \( S_\infty \), is that \( C \) is always a subset of \( S_\infty \). This is because in order to use \( B \) for our safety objective (2), the control invariant set \( C \) should be a subset of \( L \), as shown in Fig. 1. Since \( S_\infty \) is the maximal control invariant set in \( L \), \( C \subseteq S_\infty \).

Also, note that \( V_\infty \) satisfies the CBF condition (9) everywhere the gradient \( \nabla_x V_\infty \) exists, for any extended class \( K_\infty \) function \( \alpha \). This is because,

\[
\max_{u \in U} D_x V_\infty(x) \cdot f(x, u) \geq 0 \geq -\alpha(V_\infty(x)).
\]

This implies that if \( V_\infty \) is differentiable in \( S_\infty \), then setting \( B = V_\infty \) works as a valid CBF with \( C = S_\infty \). However, if it is not the case, it is hard to devise a CBF such that its zero-superlevel set recovers the maximal control invariant set in \( L \) without relaxing its differentiability condition. Note that choosing \( B = l \), which makes \( C = L \), would not be a valid CBF in general. In many cases, a valid handcrafted CBF results in its zero-superlevel set \( C \) strictly smaller than \( S_\infty \).

III. ROBUST CONTROL BARRIER-VALUE FUNCTION AND HAMILTON-JACOBI-BASED VERIFICATION

Note that the condition the CBF-based safe control should satisfy, \( D_x B(x) \cdot f(x, u) \geq -\alpha(B(x)) \), from Theorem 1, is less restrictive than the condition the optimal control from \( V \) should satisfy, \( \min_{d \in D} D_x V(x, s) \cdot f(x, u, d) \geq 0 \). This is mainly because of the introduction of \( -\alpha(\cdot) \) on the right hand side of (9). Inspired by this and the fact that when \( \alpha(B(x)) = \gamma B(x) \), \( \gamma \) serves as the maximal discount rate of \( B \), we define the following new value function.

**Definition 2.** A Robust Control Barrier-Value Function (CBVF) \( B_\gamma : \mathbb{R}^n \times (-\infty, 0] \to \mathbb{R} \) is defined as

\[
B_\gamma(x, t) := \min_{\xi \in \Xi(t, o)} \max_{u \in U(t, o)} \min_{s \in [t, 0]} e^{\gamma(s-t)} l(x(s)), \quad (11)
\]

where \( x(\cdot) \) solves for \( x(t, t, u, \xi)[u] \), for some \( \gamma \geq 0 \) and \( \forall t \leq 0 \). At \( t = 0 \), we get terminal condition \( B_\gamma(x, 0) = l(x) \).

Note that \( B_\gamma \) is defined for each fixed value of \( \gamma \geq 0 \). Now, consider the case \( \gamma = 0 \). For this case, the definition of \( B_0 \) in (11) matches with the definition of the original reachability-based value function in (4). This is not surprising because (11) should be regarded as a special case of the reachability problem, whose target function is exponentially decaying backward in time.

Since (11) is an optimal control problem under a differential game setting, Bellman’s principle of optimality can be applied to derive the dynamic programming principle for \( B_\gamma \).

**Theorem 2.** (Dynamic Programming Optimality Condition) For the Robust CBVF \( B_\gamma \) in Definition 2, for each \( t < t + \delta \leq 0 \), the following is satisfied.

\[
B_\gamma(x, t) = \min_{\xi \in \Xi(t, o)} \max_{u \in U(t, o)} \{ \min_{s \in [t, t+\delta]} e^{\gamma(s-t)} l(x(s)), e^{\gamma \delta} B_\gamma(x(t+\delta), t+\delta) \} \quad (12)
\]

where \( x(\cdot) \) solves for \( (x, t, u, \xi_d) \).

**Proof.** See Appendix.

Theorem 2 leads to the derivation of the following theorem, which is the main theoretical result of this paper, showing that \( B_\gamma \) can be obtained by solving a particular variational inequality that has the form of HJI-VI.

**Theorem 3.** The Robust CBVF \( B_\gamma \) is a Lipschitz continuous unique viscosity solution of the variational inequality below, called the CBVF-VI with the terminal condition \( B_\gamma(x, 0) = l(x) \):

\[
0 = \min_{u \in U} \left\{ l(x) - B_\gamma(x, t), D_t B_\gamma(x, t) + \max_{d \in D} D_x B_\gamma(x, t, d) \cdot f(x, u, d) + \gamma B_\gamma(x, t) \right\}. \quad (13)
\]

**Proof.** See Appendix.

The following proposition shows that like the original reachability-based value function \( V \) from (4), \( B_\gamma \) can also be used to verify the viability kernel \( S(t) \). In other words, the zero-superlevel set of the Robust CBVF contains every initial state from which robust safety guarantee is possible for a chosen time span. This is in sharp contrast to the CBFs, since the safe invariant set from a given CBF is only guaranteed to be a subset of the maximal control invariant set. Moreover, since CBVF is concerned with safety for finite-time horizon, the obtained safe set can be much bigger than the control invariant set from CBFs. Therefore, in addition to the fact that the CBVF is constructive, the main benefit of using the CBVF is that it recovers the biggest permissible region for the system for maintaining safety (Fig. 1).

**Proposition 2.** For each \( t \leq 0 \), define \( C_\gamma(t) := \{ x \in \mathbb{R}^n : B_\gamma(x, t) \geq 0 \} \). Then, \( \forall t \leq 0 \), \( C_\gamma(t) = S(t) \).

**Proof.** For each \( t \in (-\infty, 0] \), consider \( x \) such that \( B_\gamma(x, t) \geq 0 \). For \( \forall \xi_d \in \Xi(t, o) \), there exists \( u \in U(t, o) \) such that \( \min_{u \in U(t, o)} e^{\gamma(s-t)} l(x(s)) \geq 0 \). Therefore, \( x \) belongs to \( S(t) \).

Consider \( x \in S(t) \). For all \( \xi_d \in \Xi(t, o) \), there exists \( u \in U(t, o) \) such that \( l(x(s)) \) is non-negative for all \( s \in [t, 0] \). Thus, \( \max_{u \in U(t, o)} \min_{u \in U(t, o)} e^{\gamma(s-t)} l(x(s)) \) is non-negative for all \( \xi_d \), and \( B_\gamma(x, t) \geq 0 \).

Finally, since \( V \) can be used to verify the viability kernel \( S(t) \), readers might wonder the additional benefit of introducing \( B_\gamma \). In the next section, we explain why using \( B_\gamma \) would be preferable to using the original value function \( V \).
IV. Optimal Control Policy of the CBVF

A. Evaluation of the optimal control policy of the CBVF

The main benefit of using the optimal controller from the new formulation of CBVF \( B_r \) instead of the original reachability-based optimal controller \( \pi^*_V \) is that it can significantly reduce the conservativeness of \( \pi^*_V \) (Remark 2).

First recall how the optimal policy \( \pi^*_V \) of \( V \) is verified: 1) when \( V(x, t) < l(x) \), it is determined by (6), and 2) when \( V(x, t) = l(x) \), any element of (7) is optimal.

From the CBVF-VI (13), we can verify the optimal control policy with respect to \( B_r \) similarly. For the first case, when \( B_r(x, t) < l(x) \), the second term of (13) must be zero; therefore the optimal control must be given by

\[
\pi^*_B(x, t) = \arg \max_{u \in U} \min_{d \in D} D_x B_r(x, t) \cdot f(x, u, d),
\]

(14)

which is similar to the first case of \( \pi^*_V \). Also, the CBVF-VI (13) implies that for this case,

\[
D_t B_r(x, t) + \min_{d \in D} D_x B_r(x, t) \cdot f(x, \pi^*_B(x, t), d) + \gamma B_r(x, t) = B_r(x, t) + \gamma B_r(x, t) = 0.
\]

(15)

For the second case, when \( B_r(x, t) = l(x) \), any element of \( K_{B_r}(x, t) := \{u \in U : D_t B_r(x, t) + \min_{d \in D} D_x B_r(x, t) \cdot f(x, u, d) + \gamma B_r(x, t) \geq 0\} \)

(16)

is optimal with respect to \( B_r \) and can be used as \( \pi^*_B(x, t) \). For this case, \( K_{B_r}(x, t) \) is always non-empty because the second term of (13) is greater or equal to 0, and for any \( u \in K_{B_r}(x, t) \) and any \( d \in D \),

\[
l(x(t)) = B_r(x(t), t) \geq -\gamma B_r(x(t), t) = -\gamma l(x(t)),
\]

(17)

where \( x(\cdot) \) solves (1) for \((x, t, u, d)\).

It is crucial to note the difference between (8) and (17). Speaking informally, both second cases of the optimal control policies with respect to \( V \) and \( B_r \) occur when the state is not safety-critical, therefore, the user is allowed to choose any \( u \) from \( K_V \) and \( K_{B_r} \) as \( \pi^*_V \) and \( \pi^*_B \), respectively. However, as Remark 2 explains, \( \pi^*_V \) still allows the state to get closer to the safety boundary. On the other hand, \( \pi^*_B \) allows \( l \) to decrease as long as it satisfies (17), which is a very similar property that CBFs have. Therefore, \( \pi^*_B \) allows for more control authority than \( \pi^*_V \), while achieving the same safety objective.

This benefit of \( B_r \) over \( V \) can be regarded as CBF’s property of becoming less conservative instilled in the HJ reachability formulation. In the next section, we will numerically demonstrate that the optimal trajectories from \( \pi^*_B \) actually behave less conservative than the optimal trajectories from \( \pi^*_V \), especially with higher value of \( \gamma \).

B. Online optimal policy synthesis for control-affine systems

We end this section by proposing a specific way of synthesizing \( \pi^*_B \) for systems affine in control and disturbance:

\[
\dot{x}(s) = f(x(s), u(s), d(s)) = p(x(s)) + q(x(s))u(s) + r(x(s))d(s),
\]

(18)

where \( p: \mathbb{R}^n \to \mathbb{R}^n \), \( q: \mathbb{R}^n \to \mathbb{R}^{n \times m} \), and \( r: \mathbb{R}^n \to \mathbb{R}^{n \times w} \).

Note that \( u = \pi^*_B(x, t) \) should satisfy

\[
D_t B_r(x, t) + \min_{d \in D} D_x B_r(x, t) \cdot f(x, u, d) + \gamma B_r(x, t) \geq 0
\]

from (15) and (16). Similarly to the CBF-QP for control-affine systems, we can incorporate this as a linear inequality constraint in a min-norm optimization based controller. When the input bound \( U \) is polytopic, the optimization becomes a QP as well:

**Robust CBVF-QP:**

\[
\pi_{QP}(x, t) = \arg \min_{u \in U} \left( u - u_{ref} \right)^T \left( u - u_{ref} \right)
\]

s.t. \( d(x(t) + D_x B_r(x, t) \cdot q(x)u + \gamma B_r(x, t) \geq 0, \)

where \( a(x(t)) = D_t B_r(x, t) + D_x B_r(x, t) \cdot p(x) + \min_{d \in D} D_x B_r(x, t) \cdot r(x)d. \)

(19c)

Note that a similar formulation is proposed in a previous work that introduces a concept of Robust CBF [25].

**Proposition 3.** For the Robust CBVF \( B_r \) and for the system (18) with linear control bound \( U \), the Robust CBVF-QP (19) is feasible everywhere \((x, t) \in \mathbb{R}^n \times (-\infty, 0]\) where the gradient \( D_x B_r(x, t) \) exists, and its solution is always an optimal policy with respect to \( B_r \).

**Proof.** For the first case, when \( B_r(x, t) < l(x) \), the constraint of the QP (19b) is satisfied but only under the equality condition since

\[
D_t B_r(x, t) + \min_{u \in U} d \in D \cdot D_x B_r(x, t) \cdot f(x, u, d) + \gamma B_r(x, t) = 0
\]

from (15). Any \( u \in U \) that satisfies the equality condition is \( \pi^*_B \). For the second case, when \( B_r(x, t) = l(x) \), \( K_{B_r} \) is exactly the feasible set of the Robust CBVF-QP.

**Remark 3.** Note that any reference control signal \( u_{ref} \) can be used in (19), since Proposition 3 holds for every feasible solution. Therefore, (19) is not only an optimal controller for \( B_r \), it also can be used as a safety filter for any kind of performance controller. As we explained in Sec. IV-A, this new safety filter is much less restrictive than the original optimal control policy of \( V \). Also, compared to applying a least-restrictive safety filter explained in Remark 2 which utilizes value function only at the boundary, the filter (19) can be applied globally inside \( S(t) \), and the optimization automatically adjusts \( u_{ref} \) to make it safe.

**Remark 4.** When the differential \( D_x V \) or \( D_x B_r \) does not exist, since \( V \) and \( B_r \) are Lipschitz continuous, either one of superdifferential or subdifferential always exists. The optimal control is determined by the same rule (14) or (16) where the differential \( D_x B_r(x, t) \) is replaced by the superdifferential \( D_x \varphi(x, t) \notin D_x B_r(x, t) \) or subdifferential \( D_x \varphi(x, t) \in D_x B_r(x, t) \) [26, Ch.3.2.5].

V. Numerical Examples

A. Double Integrator Example

The running example in this subsection will be a simple 2D double integrator. Its system dynamics are \( \dot{z} = v + d \), \( \dot{v} = u \), with states position \( z \) and velocity \( v \), disturbance \( d \in [-0.2, 0.2] \) and control \( u \in [-1, 1] \). Figure 2 shows a comparison of the functions, zero level sets, trajectories, and control signals for three different values of \( \gamma \). On the top in
This paper introduces the notion of a Control Barrier-Value Function (CBVF) and provides a method for its construction based on HJ reachability analysis. This CBVF produces the maximal safe set for a desired safety constraint, and can handle bounded control and disturbances. We also introduce the Robust CBVF-QP for online control, and demonstrate our findings in simulation on a double integrator and dubins car. We believe that CBVF-QPs will be of use to both the reachability and CBF communities, and look forward to using it in many applications. Also, we look forward to extending this analysis to Control Lyapunov Functions, similarly to [29].

**VI. Conclusion**

The CBVF-QP maintains safety, however, because of its safety concern for infinite-time horizon, the system is unable to reach the goal. On the bottom, the time-varying CBVF-QP allows the system to safely reach the goal within the finite-time horizon. This formulation can be used for scenarios that require safety only for a fixed time [27], [28], for example a hybrid system that requires the system to only stay safe until it reaches the goal like legged robots.

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Appendix

The following proofs of Theorem 2 and Theorem 3 inherit the structure from the standard proof of viscosity solution of HJI Partial Differential Equation (HJI-PDE) for differential games [20]. Note that the proofs hold for compact $U, D$ without convexity condition. Here, we use notation $\psi_{x,t}^{u,d} \equiv x : [t,0] \to \mathbb{R}^n$, where $x(\cdot)$ solves (1) for $(x, t, u, d)$, instead of $x(\cdot)$, to specify control and disturbance signal in a convenient way. We use $\Xi_t \equiv \Xi_{[t,0]}, U_t \equiv U_{[t,0]}$.

Proof. (Proof of Theorem 2)

Define $W(x,t)$ as the right hand side of (12). For $\forall \varepsilon > 0$, $\exists \eta \in \Xi_t$ such that $\forall u \in U_t$,

$$W(x,t) \geq \min \left\{ \inf_{s \in [t,t+\delta]} e^{\gamma(s-t)}l(\psi_{x,t}^{u,n}[u](s)), e^{\gamma t} B_\gamma (\psi_{x,t}^{u,n}[u](t+\delta), t+\delta) \right\} - \varepsilon \quad (20)$$

From the definition of $B_\gamma$ for each $y \in \mathbb{R}^n$,

$$B_\gamma(y,t+\delta) = \inf_{x \in \Xi_t} \sup_{u \in U_t} \inf_{s \in [t,t+\delta]} e^{\gamma(s-t)}l(\psi_{y,t+\delta}^{u,n}[u](s)).$$

Therefore, $\forall \varepsilon > 0$, $\exists \eta \in \Xi_{t+\delta}$ such that for $\forall u \in U_{t+\delta}$,

$$B_\gamma(y,t+\delta) \geq \inf_{s \in [t,t+\delta]} e^{\gamma(s-t)}l(\psi_{x,t}^{u,n}[u](s)) - \varepsilon.$$

With $y := \psi_{x,t}^{u,n}[u](t+\delta)$, define

$$\xi_d[u] := \left\{ \begin{array}{ll} \eta[u](s) & \text{for } t \leq s \leq t+\delta \\ \eta[u](s) & \text{for } t+\delta < s \leq 0 \end{array} \right.$$ 

Then, from (20) and (21), for $\forall u \in U_t$,

$$W(x,t) \geq \inf_{s \in [t,t+\delta]} e^{\gamma(s-t)}l(\psi_{x,t}^{u,n}[u](s)) - 2\varepsilon \quad \text{by taking } \varepsilon_1 = -e^{-\gamma \delta}.$$

Therefore,

$$W(x,t) \geq B_\gamma(x,t) - 2\varepsilon \quad \forall \varepsilon > 0. \quad (22)$$

On the other hand, by definitions of $B_\gamma$ and $W$, for $\forall \varepsilon > 0$, $\exists \eta \in \Xi_t$ such that for $\forall u \in U_t$,

$$\inf_{s \in [t,t+\delta]} e^{\gamma(s-t)}l(\psi_{x,t}^{u,n}[u](s)) \leq B_\gamma(x,t) + \varepsilon \quad (23)$$

$$W(x,t) \leq \sup_{u \in U_t} \min \left\{ \inf_{s \in [t,t+\delta]} e^{\gamma(s-t)}l(\psi_{x,t}^{u,n}[u](s)), e^{\gamma t} B_\gamma (\psi_{x,t}^{u,n}[u](t+\delta), t+\delta) \right\} \quad (24)$$

Therefore, $\exists u_0 \in U_t$ such that

$$W(x,t) \leq \min \left\{ \inf_{s \in [t,t+\delta]} e^{\gamma(s-t)}l(\psi_{x,t}^{u_0,n}[u_0](s)), e^{\gamma t} B_\gamma (\psi_{x,t}^{u_0,n}[u_0](t+\delta), t+\delta) \right\} + \varepsilon$$

For $\forall u \in U_{t+\delta}$, define

$$\tilde{u}(s) := \left\{ \begin{array}{ll} u_0(s) & \text{for } t \leq s \leq t+\delta \\ u(s) & \text{for } t+\delta < s \leq 0 \end{array} \right.$$ 

and define $\tilde{\eta} \in \Xi_{t+\delta}$ by $\tilde{\eta}[u](s) = \eta[u](s)$ for $s \in [t,t+\delta]$. Then, with $y := \psi_{x,t}^{u_0,n}[u](t+\delta)$, by definition of $B_\gamma$,

$$B_\gamma(y,t+\delta) \leq \sup_{u \in U_{t+\delta}} \inf_{s \in [t,t+\delta]} e^{\gamma(s-t)}l(\psi_{x,t}^{u_0,n}[u](s)).$$

Therefore, $\forall \varepsilon > 0$, $\exists u_1 \in U_{t+\delta}$ such that

$$B_\gamma(y,t+\delta) \leq \inf_{s \in [t,t+\delta]} e^{\gamma(s-t)}l(\psi_{x,t}^{u_0,n}[u_0](s)) + \varepsilon.$$ 

Therefore, from (23),

$$W(x,t) \leq B_\gamma(x,t) + 3\varepsilon \quad \forall \varepsilon > 0. \quad (26)$$

The proof is done from (22) and (26).

Proof. (Proof of Theorem 3)

According to the definition of viscosity solution [26], Theorem 3 is equivalent to $B_\gamma$ satisfying the following statements.

1) For $\forall \varphi(x,t) \in C^1(\mathbb{R}^n \times (-\infty,0])$ such that $B_\gamma - \varphi$ has a local maximum $0$ at $(x_0,t_0) \in \mathbb{R}^n \times (-\infty,0]$,

$$0 \leq \min \left\{ l(x_0) - \varphi(x_0,t_0), \right\}$$

$$D_l \varphi(x_0,t_0) + \max_{u \in U_{t_0}} \min_{d \in D} \left( D_d \varphi(x_0,t_0) \cdot f(x_0,u,d) + \gamma \varphi(x_0,t_0) \right).$$

2) For $\forall \varphi(x,t) \in C^1(\mathbb{R}^n \times (-\infty,0])$ such that $B_\gamma - \varphi$ has a local minimum $0$ at $(x_0,t_0) \in \mathbb{R}^n \times (-\infty,0]$,

$$0 \geq \min \left\{ l(x_0) - \varphi(x_0,t_0), \right\}$$

$$D_l \varphi(x_0,t_0) + \max_{u \in U_{t_0}} \min_{d \in D} \left( D_d \varphi(x_0,t_0) \cdot f(x_0,u,d) + \gamma \varphi(x_0,t_0) \right).$$

We use the following lemma to prove 1) and 2).

Lemma 1. For $\varphi(x,t) \in C^1(\mathbb{R}^n \times (-\infty,0])$, define $A_\varphi(x,t,u,d) := D_l \varphi(x,t) + D_d \varphi(x,t) \cdot f(x,u,d) + \gamma \varphi(x,t).$

(a) If $\exists \theta > 0$, $\exists (x_0,t_0) \in \mathbb{R}^n \times (-\infty,0]$ such that $\max_{u \in U_{t_0}} \min_{d \in D} A_\varphi(x_0,t_0,u,d) \leq -\theta$, there exists a small enough $\delta > 0$, $\exists \xi_d \in \Xi_{t_0}$ such that $\forall u \in U_{t_0}$,

$$e^{\gamma t} \varphi(\psi_{x,t_0}^{u_0} \xi_d[t+\delta], t+\delta) - \varphi(x_0,t_0) \leq -\frac{\theta}{2} \delta. \quad (30)$$

(b) If $\exists \theta > 0$, $\exists (x_0,t_0) \in \mathbb{R}^n \times (-\infty,0]$ such that $\max_{u \in U_{t_0}} \min_{d \in D} A_\varphi(x_0,t_0,u,d) \geq \theta$, there exists a small enough $\delta > 0$, $\exists \xi_d \in \Xi_{t_0}$, $\exists u \in U_{t_0}$ such that

$$e^{\gamma t} \varphi(\psi_{x,t_0}^{u_0} \xi_d[t+\delta], t+\delta) - \varphi(x_0,t_0) \geq \frac{\theta}{2} \delta. \quad (31)$$

Lemma 1 is a modification of [20, Lemma 4.3.] for general HJI-PDE to CBVF-VI. For its proof, please refer to [20].

Proof of 1). Let (27) be false. Then one of the followings should hold.

$$\exists \theta_1 > 0 \text{ s.t. } l(x_0) - \varphi(x_0,t_0) \leq -\theta_1 \quad (32a)$$

$$\exists \theta_2 > 0 \text{ s.t. } D_l \varphi(x_0,t_0) + \max_{u \in U_{t_0}} \min_{d \in D} A_\varphi(x_0,t_0,u,d) + \gamma \varphi(x_0,t_0) \geq -\theta_2 \quad (32b)$$

If (32a) is true, by continuity of $l$ and $\psi$, $\exists \delta > 0$ such that for all $u \in U_{t_0}$, $\xi_d \in \Xi_{t_0}$, $s \in [t_0,t_0+\delta]$,

$$\left| e^{\gamma(s-t_0)}l(\psi_{x,t_0}^{u_0} \xi_d[t], s) - l(x_0) \right| \leq \frac{\theta_1}{2}. \quad (33)$$

$$\Rightarrow e^{\gamma(s-t_0)}l(\psi_{x,t_0}^{u_0} \xi_d[t], s) \leq l(x_0) + \frac{\theta_1}{2} = B_\gamma(x_0,t_0) - \frac{\theta_1}{2}.$$
Plugging this into the dynamic programming principle (12),
\[
B_\gamma(x_0, t_0) \leq \inf_{\xi \in U_0} \sup_{t_0, t_0 \in [0, t_0]}\inf_{s \in [t_0, t_0]} e^{\gamma(s-t_0)} I(x_0, t_0, u(s)) \leq B_\gamma(x_0, t_0) - \frac{\theta_1}{2}.
\]
This is a contradiction, therefore, (32a) is false.

Next, if (32b) is true, from Lemma 1.a, for small enough \( \delta > 0 \), \( \exists \eta \in \Sigma_{t_0} \) such that for all \( u \in U_0 \),
\[
e^{-\delta} \varphi(x_{t_0, t_0}^{u, \eta}(t_0, t_0), t_0, t_0) - \varphi(x_0, t_0) \leq -\frac{\theta_2}{2} \delta.
\]
Since \( B_\gamma \) is negative local maximum 0 at \( (x_0, t_0) \),
\[
0 = \gamma \delta B_\gamma(x_{t_0, t_0}^{u, \eta}(t_0, t_0), t_0, t_0) \leq e^{-\delta} \varphi(x_{t_0, t_0}^{u, \eta}(t_0, t_0), t_0, t_0) + \frac{\theta_2}{2} \delta.
\]
Finally, (from 12), we get,
\[
B_\gamma(x_0, t_0) \leq \sup_{u \in U_0} \min \left\{ \varepsilon(x_0, t_0, \xi_{t_0}) \mid e^{\gamma(s-t_0)} I(x_0, t_0, u(s)), e^{-\delta} B_\gamma(x_{t_0, t_0}^{u, \xi}(t_0, t_0), t_0, t_0) \right\}
\]
which is a contradiction. Therefore, (32b) is false. \( \square \)

**Proof of 2.** Let (28) be false. Then both of the followings should hold.
\[
(33\text{a}) \quad \exists \theta_1 > 0 \text{ s.t. } I(x_0) - \varphi(x_0, t_0) \geq \theta_1
\]
\[
(33\text{b}) \quad \exists \theta_2 > 0 \text{ s.t. } D_\gamma \varphi(x_0, t_0) = \max \min_{u \in U} D_\varepsilon \varphi(x_0, t_0) \cdot f(x_0, u, d)
\]
\[
+ \frac{\gamma}{\gamma} \varphi(x_0, t_0) \geq \theta_2
\]
From (33a), by continuity of \( I \) and \( \psi \), \( \exists \xi \delta > 0 \) such that for all \( u \in U_0 \), \( \xi_{t_0} \in \Sigma_{t_0} \), \( s \in [t_0, t_0 + \delta] \),
\[
0 = \gamma \delta I(x_{t_0, t_0}^{u, \xi}(t_0, t_0), t_0, t_0) - I(x_0) \leq \frac{\theta_1}{2}
\]
Finally, (from 12), we get,
\[
0 = \gamma \delta B_\gamma(x_{t_0, t_0}^{u, \xi}(t_0, t_0), t_0, t_0) \leq e^{-\delta} \varphi(x_{t_0, t_0}^{u, \xi}(t_0, t_0), t_0, t_0) + \frac{\theta_2}{2} \delta.
\]
This is a contradiction. \( \square \)

Note that since \( B_\gamma \) satisfies both 1) and 2), uniqueness and Lipschitz continuity of \( B_\gamma \) can be derived similarly to [30, Th.4.2] and [20, Th.3.2.2., respectively.


