Geometric Control of Cooperating Multiple Quadrotor UAVs with a Suspended Payload

Taeyoung Lee*, Koushil Sreenath, and Vijay Kumar

***Abstract*— This paper investigates tracking controls for an arbitrary number of cooperating quadrotor unmanned aerial vehicles with a suspended load. Assuming that a point mass is connected to multiple quadrotors by rigid massless links, control systems for quadrotors are constructed such that the point mass asymptotically follows a given desired trajectory and quadrotors maintain a prescribed formation, either relative to the point mass or with respect to the inertial frame. These are developed in a coordinate-free fashion to avoid singularities and complexities associated with local parameterizations. The desirable features are illustrated by several numerical examples, including a flying inverted spherical pendulum on a quadrotor.**

I. Introduction

Quadrotor unmanned aerial vehicles (UAV) have been envisaged for various applications such as surveillance or mobile sensor networks as well as for educational purposes. In particular, nonlinear control systems for complex maneuvers of quadrotors have been studied, and aggressive maneuvers are demonstrated experimentally by utilizing the high thrust-to-weight ratio of quadrotors [1], [2], [3].

These properties of quadrotors are also desirable for load carrying and transportation. Small-size single or multiple autonomous vehicles are considered for load transportation and deployment [4], [5], [6]. Nonlinear tracking control systems are developed for a single quadrotor UAV with a cable-suspended load in [7] and a companion paper [8].

Load transportation with multiple quadrotors is useful when the load is heavy compared with the maximum thrust of a single quadrotor, or when additional redundancy is required for safety. But, this is challenging since *dynamically coupled* quadrotors should cooperate safely to transport load. This is in contrast to the existing results on formation control of decoupled multi-agent systems.

In this paper, we consider an arbitrary number of quadrotors that are connected to a point mass via rigid links. The equations of motion are derived from the variational principle, and control systems are developed such that the point mass asymptotically follows a given smooth desired trajectory. Two formation flight modes are introduced to control the formation of quadrotors with respect to the point mass. In the existing control systems for a load-carrying quadrotor, such as [4], a quadrotor is designed to follow pre-computed minimum swing trajectories while rejecting the force and moment exerted by the load that are considered as disturbances. The control systems proposed in this paper explicitly consider the coupling effects between the load dynamics and the dynamics of multiple quadrotors for safe transportation along complex trajectories.

Another distinct feature is that the equations of motion and the control systems are developed directly on the nonlinear configuration manifold in a coordinate-free fashion. This yields remarkably compact expressions for the dynamic model and controllers, compared with local coordinates that often require symbolic computational tools due to complexity of multibody systems. Furthermore, singularities of local parameterization are completely avoided to generate agile maneuvers in a uniform way.

If the links are assumed to be rigid, the proposed control system is also applied to the cases where selected quadrotors are below the load, to obtain so-called *flying inverted pendulum*. Linear control systems have been developed and implemented to stabilize few selected nominal trajectories in [9]. The proposed control system is based on the full nonlinear dynamics, and it guarantees exponential stability of flying inverted pendulum for arbitrary desired trajectories.

Compared with the companion paper [8] that is focused on tracking, differential flatness, and experiments of a single quadrotor with a suspended load, this paper proposes a cooperative framework of an arbitrary number of quadrotors. Due to page limit constraint, all of the proofs are relegated to [10].

II. Dynamics Model

Consider n quadrotor UAVs that are connected to a point mass m_y via massless links, as illustrated at Fig. 1. We choose an inertial reference frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and body-fixed frames $\{\vec{b}_{i_1}, \vec{b}_{i_2}, \vec{b}_{3_i}\}$ for $i \in \mathcal{I} \triangleq \{1, \dots n\}$. Throughout this paper, the subscript i is assumed to be an element of the index set $\mathcal{I}$. The origin of the i-th body-fixed frame is located at the center of mass of the i-th quadrotor. The third body-fixed axis $\vec{b}_{i_3}$ is normal to the plane defined by the centers of rotors, and it points downward.

The location of the point mass m_y in the inertial frame is denoted by $y \in \mathbb{R}^3$. The direction of the i-th link from the i-th quadrotor toward the point mass is defined as $q_i \in \mathsf{S}^2$, where $\mathsf{S}^2 = \{q \in \mathbb{R}^3 \,|\, \|q\| = 1\}$. The attitude of the i-th quadrotor is denoted by $R_i \in \mathsf{SO(3)} = \{R \in \mathbb{R}^{3\times 3} \,|\, R^T R = I, \det[R] = 1\}$, which is the linear transformation of the

Taeyoung Lee, Mechanical and Aerospace Engineering, George Washington University, Washington, DC 20052 tylee@gwu.edu

Koushil Sreenath, Mechanical Engineering, Carnegie Mellon University, Pittsburgh, PA 15213 koushils@cmu.edu

Vijay Kumar, GRASP Lab, University of Pennsylvania, Philadelphia, PA 19104 kumar@seas.upenn.edu

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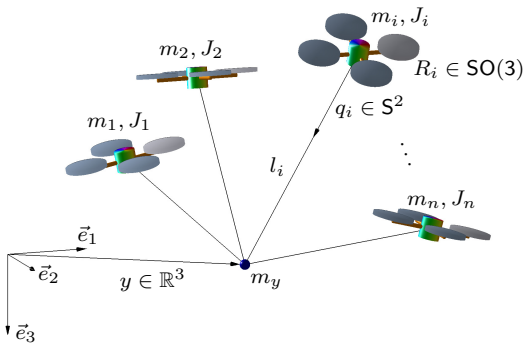


Fig. 1. Dynamics model: n quadrotors are connect to a point mass m_y via massless links l_i. The corresponding configuration manifold is $\mathbb{R}^3 \times (\mathsf{S}^2 \times \mathsf{SO}(3))^n$.

representation of a vector from the i-th body-fixed frame to the inertial frame. Let $l_i \in \mathbb{R}$ be the length of the i-th link, and it is assumed that each link is attached to the mass center of the corresponding quadrotor. Then, the location of the mass center of the i-th quadrotor, namely $x_i \in \mathbb{R}^3$ is given by $x_i = y - l_i q_i$. The corresponding configuration manifold of this system is $\mathbb{R}^3 \times (\mathsf{S}^2 \times \mathsf{SO}(3))^n$.

The dynamic model of each quadrotor is identical to [1]. The mass and the inertia matrix of the i-th quadrotor are denoted by $m_i \in \mathbb{R}$ and $J_i \in \mathbb{R}^{3\times 3}$, respectively. The i-th quadrotor can generates a thrust $-f_i R_i e_3 \in \mathbb{R}^3$ with respect to the inertial frame, where $f_i \in \mathbb{R}$ is the total thrust magnitude and $e_3 = [0, 0, 1]^T \in \mathbb{R}^3$. It also generates a moment $M_i \in \mathbb{R}^3$ with respect to its body-fixed frame. The control input of this system corresponds to $\{f_i, M_i\}_{i\in\mathcal{I}}$.

Throughout this paper, the 2-norm of a matrix A is denoted by $\|A\|$, and the dot product is denoted by $x \cdot y = x^T y$.

A. Equations of Motion

The kinematic equations for the direction of the i-th link, and the attitude of the i-th quadrotor are given by

$$\dot{q}_i = \omega_i \times q_i = \hat{\omega}_i q_i, \tag{1}$$

$$\dot{R}_i = R_i \hat{\Omega}_i, \tag{2}$$

where $\omega_i \in \mathbb{R}^3$ is the angular velocity of the i-th link, satisfying $q_i \cdot \omega_i = 0$, and $\Omega_i \in \mathbb{R}^3$ is the angular velocity of the i-th quadrotor expressed with respect to its body-fixed frame. The *hat map* $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is defined by the condition that $\hat{x}y = x \times y$ for all $x, y \in \mathbb{R}^3$, and the inverse of the hat map is denoted by the *vee map* $\vee : \mathfrak{so}(3) \to \mathbb{R}^3$.

The velocity of the i-th quadrotor is given by $\dot{x}_i = \dot{y} - l_i \dot{q}_i$. The kinetic energy of the system is composed of the translational kinetic energy of the point mass, and the translational and rotational kinetic energy of quadrotors:

$$\mathcal{T} = \frac{1}{2} m_y \|\dot{y}\|^2 + \sum_{i=1}^{n} \frac{1}{2} m_i \|\dot{y} - l_i \dot{q}_i\|^2 + \frac{1}{2} \Omega_i \cdot J_i \Omega_i. \tag{3}$$

The gravitational potential energy is given by

$$\mathcal{U} = -m_y g e_3 \cdot y - \sum_{i=1}^{n} m_i g e_3 \cdot (y - l_i q_i), \tag{4}$$

where it is assumed that the unit-vector e_3 points downward along the gravitational acceleration as shown at Fig. 1. The corresponding Lagrangian of the system is $\mathcal{L} = \mathcal{T} - \mathcal{U}$.

Coordinate-free form of Lagrangian mechanics on the two-sphere S^2 and the special orthogonal group $\mathsf{SO}(3)$ for various multibody systems has been studied in [11], [12]. The key idea is representing the infinitesimal variation of $q_i \in \mathsf{S}^2$ in terms of the exponential map:

$$\delta q_i = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \exp(\epsilon \hat{\xi}_i) q_i = \xi_i \times q_i,$$

for a vector $\xi_i \in \mathbb{R}^3$ with $\xi_i \cdot q_i = 0$. Similarly, the variation of R_i is given by $\delta R_i = R_i \hat{\eta}_i$ for $\eta \in \mathbb{R}^3$.

By using these expressions, the equations of motion can be obtained from Hamilton's principle as follows:

$$M_q(\ddot{y} - g e_3) = \sum_{i=1}^{n} (-m_i l_i \|\omega_i\|^2 q_i + u_i^{\parallel}), \tag{5}$$

$$\dot{\omega}_i = \frac{1}{l_i} \hat{q}_i (\ddot{y} - g e_3) - \frac{1}{m_i l_i} \hat{q}_i u_i^{\perp}, \tag{6}$$

$$J_i \dot{\Omega}_i + \Omega_i \times J_i \Omega_i = M_i, \tag{7}$$

where $M_q = m_y I + \sum_{i=1}^{n} m_i q_i q_i^T \in \mathbb{R}^{3\times 3}$, which is symmetric, positive-definite for any q_i. The vector $u_i \in \mathbb{R}^3$ represents the control force at the i-th quadrotor, i.e., $u_i = -f_i R_i e_3$, and the vectors $u_i^{\parallel}$ and $u_i^{\perp} \in \mathbb{R}^3$ denote the orthogonal projection of u_i along q_i, and the orthogonal projection of u_i to the plane normal to q_i, respectively:

$$u_i^{\parallel} = (I + \hat{q}_i^2) u_i = (q_i \cdot u_i) q_i = q_i q_i^T u_i, \tag{8}$$

$$u_i^{\perp} = -\hat{q}_i^2 u_i = -q_i \times (q_i \times u_i) = (I - q_i q_i^T) u_i. \tag{9}$$

Therefore, $u_i = u_i^{\parallel} + u_i^{\perp}$.

III. Control System Design

A. Tracking Control Problem Formulation

Suppose that a desired trajectory of the point mass, namely $y_d(t) \in \mathbb{R}^3$ is given as a smooth function of time. The goal is to design control inputs $\{f_i, M_i\}_{i\in\mathcal{I}}$ such that $y \to y_d$ as $t \to \infty$. This corresponds to a load transportation problem.

If there are multiple quadrotors, the formation of quadrotors with respect to the point mass, or equivalently the direction of links can be controlled as well. We consider two cases, namely *relative formation control* that is formulated for any number of quadrotors, and *inertial formation control* that can be formulated only if $n \geq 3$.

1) Relative Formation Control Mode ($n \geq 1$)*:* This is similar to virtual structure approaches in formation control [13], [14], where the desired motion for quadrotors in formation is described as a motion of a virtual rigid body rotating about the point mass. The relative configuration between quadrotors is controlled, but the orientation of the formation with respect to the inertial frame may change.

In this case, a relative configuration frame $\{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ is introduced, and the desired relative configuration between quadrotors is specified by $\{r_{i_d} \in \mathsf{S}^2\}_{i\in\mathcal{I}}$ with respect to the relative configuration frame. More explicitly, it is desired

that $q_i \to q_{i_d} \triangleq Qr_{i_d}$ as $t \to \infty$ for all $i \in \mathcal{I}$, for an rotation matrix $Q \in \mathsf{SO}(3)$ representing the transformation from the relative configuration frame to the inertial frame. Consequently, the j-th column of the matrix Q corresponds to the direction of $\vec{c}_j$ represented with respect to the inertial frame for $j \in \{1, 2, 3\}$. As this is to specify the *relative* configuration between quadrotors, the rotation matrix Q is completely arbitrary, and later it is designed to follow the given command y_d for the load.

Without loss of generality, it is assumed that

$$r_{1_d} \cdot e_2 = 0, \quad \frac{\sum_{i=1}^n r_{i_d}}{\| \sum_{i=1}^n r_{i_d} \|} = e_3, \tag{10}$$

where $e_2 = [0, 1, 0]^T$, $e_3 = [0, 0, 1]^T \in \mathbb{R}^3$. These state that the desired direction of the first link q_{1_d} lies in the plane spanned by $\vec{c}_1$ and $\vec{c}_3$, and the sum of the desired direction vectors q_{i_d} is parallel to $\vec{c}_3$. There is no restriction caused by (10), as the orientation of the relative configuration frame or Q is arbitrary. But, it is assumed that

$$r_{i_d} \cdot e_3 \neq 0, \quad \text{for every } i \in \mathcal{I}. \tag{11}$$

The above equation states that q_{i_d} is not perpendicular to $\vec{c}_3$. This assumption is required since the third-axis $\vec{c}_3$ of the relative configuration frame is chosen as the direction of the desired resultant force later, and if q_{i_d} is normal to $\vec{c}_3$ then it cannot contribute to the desired resultant force at all.

Note that when $n = 1$, the desired relative configuration satisfying (10), (11) is uniquely given by $r_{1_d} = e_3$. In this case, the direction of the link is completely determined by the objective to track the desired trajectory y_d.

When $n \geq 2$, after the links are converged to the desired relative configuration with respect to the point mass, there exists additional one-dimensional degree of freedom in the control system, which corresponds to the rotation of the quadrotor formation about $\vec{c}_3$. To resolve this, the desired direction of the first link, namely $s_d \in \mathsf{S}^2$, with respect to the inertial frame is introduced. The control system is chosen such that q_1 also asymptotically follows s_d. As there is only one-dimensional remaining degree of freedom in controls, we cannot guarantee that q_1 exactly follows s_d in general. Instead, q_1 asymptotically converges to the plane spanned by $\vec{c}_3$ and s_d. The detailed convergence properties are described more precisely in the subsequent developments. In short, the relative configuration between quadrotors with respect to the point mass is specified by r_{i_d} satisfying (10), (11), and the heading angle of the desired formation is specified by s_d.

2) Inertial Formation Control Mode ($n \geq 3$)*:* When there are more than two quadrotors, the desired direction of the links with respect to the inertial frame is specified by smooth curves $\{q_{i_d}(t) \in \mathsf{S}^2\}_{i \in \mathcal{I}}$. A control system is designed such that $q_i \to q_{i_d}$ as $t \to \infty$. It is assumed that at least three desired directions are mutually linearly independent for all $t \geq 0$. These two modes are compared further at Remark 1.

B. Control System Design for a Simplified Dynamics Model

At (5), (6), the total thrust of the i-th quadrotor is given by $u_i = -f_i R_i e_3$. This implies that the total thrust magnitude f_i can be arbitrarily chosen, but the thrust vector is always along the third body-fixed axis of the quadrotor. As the rotational dynamics of quadrotors given by (7) are not affected by the dynamics of the mass and the links, in this subsection, we first construct a control system only (5), (6) by assuming that u_i is a control input that can be arbitrarily chosen. The effects of the rotational dynamics of quadrotors are incorporated in the next subsection for the complete dynamics model.

The dynamics of the mass and the links given by (5), (6) have distinct features: the acceleration $\ddot{y}$ of the point mass is controlled by the parallel components $u_i^{\parallel}$ at (5), and the angular acceleration $\dot{\omega}_i$ of each link is controlled by the normal components $u_i^{\perp}$ at (6). These motivate the following controller structures. The parallel components $u_i^{\parallel}$ are designed to follow the desired trajectory y_d. This also constructs the desired direction q_{i_d} of each link for the relative formation control mode. The normal components $u_i^{\perp}$ are designed such that q_i converges to q_{i_d}.

1) Design of the Parallel Components $u_i^{\parallel}$*:* The parallel component of the control input is chosen such that

$$u_i^{\parallel} = m_i l_i \|\omega_i\|^2 q_i + \mu_i + \frac{m_i}{m_y} q_i q_i^T \sum_{j=1}^n \mu_j, \tag{12}$$

where $\mu_i \in \mathbb{R}^3$ is a virtual control input that is defined later at (19). It is chosen such that μ_i is parallel to q_i, and $q_i q_i^T$ at the last term corresponds to the projection along q_i. Therefore, the right hand side of (12) is also parallel to q_i. We can show that $-\mu_i$ corresponds to the force exerted by the i-th link to the i-th quadrotor for the controlled dynamics, and the corresponding tension of the i-link is given by $-q_i \cdot \mu_i$. Substituting it into (5),

$$M_q(\ddot{y} - ge_3) = (I + \sum_{i=1}^n \frac{m_i}{m_y} q_i q_i^T) \sum_{j=1}^n \mu_j = \frac{1}{m_y} M_q \sum_{j=1}^n \mu_j.$$

Multiplying both sides by M_q^{-1} and substituting it into (6), the equations of motion for y and ω_i are simplified as

$$m_y(\ddot{y} - ge_3) = \sum_{i=1}^n \mu_i, \tag{13}$$

$$\dot{\omega}_i = \frac{1}{m_y l_i} \hat{q}_i \sum_{\substack{j=1 \\ j \neq i}}^n \mu_j - \frac{1}{m_i l_i} \hat{q}_i u_i^{\perp}. \tag{14}$$

These are used for subsequent control system developments.

Define the tracking error variable $e_y \in \mathbb{R}^3$ as

$$e_y = y - y_d. \tag{15}$$

A simple PD-type control force to track the given command y_d is chosen as follows:

$$F_d = m_y(-k_y e_y - k_{\dot{y}} \dot{e}_y + \ddot{y}_d - ge_3), \tag{16}$$

where $k_y, k_{\dot{y}}$ are positive constants. It is straightforward to check that if $\sum_{i=1}^n \mu_i = F_d$ at (13), then the zero equilibrium of the tracking errors $(e_y, \dot{e}_y)$ is exponentially stable.

However, the virtual control input μ_i cannot be arbitrarily chosen as μ_i is constrained to be parallel to q_i at (13), and

the desired formation of quadrotors with respect to the point mass is also specified. Next, the virtual control input μ_i is designed to follow the given desired trajectory y_d and the desired formation for each of the relative formation control mode and the inertial formation control mode.

2) Design of μ_i for Relative Formation Control: Here, we construct the desired directions q_{i_d} of the links with respect to the inertial frame such that F_d is equal to $\sum_{i=1}^n \mu_i$ and the given desired relative configuration of quadrotors are satisfied when $q_i = q_{i_d}$. More explicitly, the rotation matrix from the relative configuration frame to the inertial frame is

$$Q = \left[-\frac{\hat{F}_d^2 s_d}{\|\hat{F}_d^2 s_d\|}, \quad -\frac{\hat{F}_d s_d}{\|\hat{F}_d s_d\|}, \quad -\frac{F_d}{\|F_d\|} \right], \tag{17}$$

where it is assumed that s_d is nonparallel with F_d. From this definition, it can be verified that $Q \in \mathsf{SO}(3)$ always. The desired direction of each link is given by

$$q_{i_d} = Q r_{i_d}. \tag{18}$$

Therefore, the given desired relative formation of quadrotors with respect to the point mass is satisfied by (18) if $q_i = q_{i_d}$. Furthermore, since $r_{1_d} \cdot e_2 = 0$, we have $Q r_{1_d} \cdot Q e_2 = q_{1_d} \cdot Q e_2 = 0$. This implies that q_{1_d} is normal to $F_d \times s_d$, i.e., q_{1_d} lies in the plane spanned by F_d and s_d. Therefore, the heading angle command is also satisfied if $q_i = q_{i_d}$.

Based on these, the virtual control input μ_i is chosen as

$$\mu_i = \frac{1}{(r_{i_d} \cdot e_3)\|\sum_{i=1}^n r_{i_d}\|}(F_d \cdot q_i) q_i. \tag{19}$$

It is designed such that the resultant control force $\sum_i \mu_i$ becomes the desired force F_d if $q_i = q_{i_d}$ as follows. From (18), (10), and (17), we can show that

$$\sum_{i=1}^n \mu_i \Big|_{q_i = q_{i_d}} = -\|F_d\| Q e_3 = F_d. \tag{20}$$

In summary, for the relative formation control mode, the parallel component of the control input is given by (12), where the virtual control input is given by (19). The desired direction of each link with respect to the inertial frame, namely q_{i_d} is also specified by (18) to satisfy the given relative formation command.

3) Design of μ_i for Inertial Formation Control ($n \geq 3$): In the inertial formation control mode, the desired direction of the links are specified by $q_{i_d}(t)$ at the problem formulation. Since $n \geq 3$, finding μ_i that is parallel to q_i such that $\sum_{i=1}^n \mu_i = F_d$ is exactly determined ($n = 3$) or underdetermined ($n > 3$). It is chosen as

$$\mu_i = [S^T (S S^T)^{-1} F_d]_i q_i, \tag{21}$$

where $S = [q_1, q_2, \ldots q_n]^{3 \times n}$, and $[x]_i$ denotes the i-th element of a vector $x \in \mathbb{R}^n$. This corresponds to the minimum-norm solution, and we can easily show that

$$\sum_{i=1}^n \mu_i = S(S^T (S S^T)^{-1} F_d) = F_d. \tag{22}$$

4) Design of the Normal Components $u_i^\perp$: For both of the relative formation control mode and the inertial formation control mode, the normal components are chosen such that $q_i \to q_{i_d}$ as $t \to \infty$ at (14). This corresponds to the tracking problem on S^2, which has been studied in [15], [16]. In [16], the direction error vector $e_{q_i} \in \mathbb{R}^3$ and the angular velocity error vector $e_{\omega_i} \in \mathbb{R}^3$ are defined as follows:

$$e_{q_i} = q_{i_d} \times q_i, \quad e_{\omega_i} = \omega_i + \hat{q}_i^2 \omega_{i_d}, \tag{23}$$

where the desired angular velocity of the i-th link is denoted by $\omega_{i_d} = q_{i_d} \times \dot{q}_{i_d} \in \mathbb{R}^3$. The normal components $u_i^\perp$ are designed such that

$$\dot{\omega}_i = -k_q e_{q_i} - k_\omega e_{\omega_i} - (q_i \cdot \omega_{i_d}) \dot{q}_i - \hat{q}_i^2 \dot{\omega}_{i_d}, \tag{24}$$

for positive constants k_q, k_ω. From (14), and using the fact that $-\hat{q}_i^2 u_i^\perp = u_i^\perp$, we obtain

$$\begin{aligned} u_i^\perp = {} & m_i l_i \hat{q}_i (-k_q e_{q_i} - k_\omega e_{\omega_i} - (q_i \cdot \omega_{i_d}) \dot{q}_i - \hat{q}_i^2 \dot{\omega}_{i_d}) \\ & - \frac{m_i}{m_y} \hat{q}_i^2 \sum_{\substack{j=1 \\ j \neq i}}^n \mu_j. \end{aligned} \tag{25}$$

The total control force is given by

$$u_i = u_i^\parallel + u_i^\perp. \tag{26}$$

The corresponding stability properties of the proposed control system for the simplified dynamics model are summarized as follows.

Proposition 1: Consider a simplified dynamics model defined by (1), (5), and (6), and two tracking control modes formulated at Section III-A. Control inputs are designed as (26), where the desired direction of links and the virtual control input for the relative formation control problem are given by (18) and (19), respectively, and the virtual control input of the inertial formation control problem is given by (21). Then, there exist controller gains $k_y, k_{\dot{y}}, k_q, k_\omega$ such that the zero equilibrium of the tracking errors $(e_y, \dot{e}_y, e_{q_i}, e_{\omega_i})$ is exponentially stable. For the relative formation control problem, the direction of the first link q_1 asymptotically converges to the plane spanned by $\ddot{y}_d - g e_3$ and s_d.

Proof: See [10]. ■

C. Control System Design for the Full Dynamics Model

The above control system for a simplified dynamics model is generalized to the full dynamics model that includes the attitude dynamics (7) of each quadrotor. The control force of the full dynamic model is given by $-f_i R_i e_3$. Here, the attitude of each quadrotors is controlled such that the direction of its third body-fixed axis becomes parallel with $-u_i$ given at (26) sufficiently fast for singular perturbation.

The construction of the attitude controller is similar with [1]. The desired direction of the third body-fixed axis of the i-th quadrotor is given by

$$b_{3_i} = -\frac{u_i}{\|u_i\|}. \tag{27}$$

There is additional one-dimensional degree of freedom corresponding to rotation about the third-body fixed axis. A

desired direction of the first body-fixed axis, namely $b_{1_i}(t) \in \mathsf{S}^2$ is introduced to resolve it. The corresponding desired attitude is chosen as

$$R_{c_i} = \left[-\frac{(\hat{b}_{3_i})^2 b_{1_i}}{\|(\hat{b}_{3_i})^2 b_{1_i}\|}, \quad \frac{\hat{b}_{3_i} b_{1_i}}{\|\hat{b}_{3_i} b_{1_i}\|}, \quad b_{3_i} \right], \tag{28}$$

and the corresponding desired angular velocity is obtained by $\Omega_{c_i} = (R_{c_i}^T \dot{R}_{c_i})^\vee \in \mathbb{R}^3$. Define the error variables for the attitude dynamics as

$$e_{R_i} = \frac{1}{2}(R_{c_i}^T R_i - R_i^T R_{c_i})^\vee, \quad e_{\Omega_i} = \Omega_i - R_i^T R_{c_i} \Omega_{c_i}.$$

The thrust magnitude and the moment vector of quadrotors are chosen as

$$f_i = -u_i \cdot R_i e_3, \tag{29}$$

$$M_i = -\frac{k_R}{\epsilon^2} e_{R_i} - \frac{k_\Omega}{\epsilon} e_{\Omega_i} + \Omega_i \times J_i \Omega_i - J_i(\hat{\Omega}_i R_i^T R_{c_i} \Omega_{c_i} - R_i^T R_{c_i} \dot{\Omega}_{c_i}), \tag{30}$$

where ϵ, k_R, k_Ω are positive constants [1].

Proposition 2: Consider the full dynamics model defined by (1), (2), (5), (6), and (7), and two tracking control modes formulated at Section III-A. Control inputs are designed as (29) and (30), where the desired control force is given by (26). Then, there exist $\epsilon^* > 0$, such that for all $\epsilon < \epsilon^*$, the zero equilibrium of the tracking errors $(e_y, \dot{e}_y, e_{q_i}, e_{\omega_i}, e_{R_i}, e_{\Omega_i})$ is exponentially stable. For the relative formation control problem, the direction of the first link q_1 asymptotically converges to the plane spanned by $\ddot{y}_d - ge_3$ and s_d.

Proof: See [10]. ■

Remark 1: When $n \geq 3$, either the relative formation control mode or the inertial formation control mode can be applied. In the relative formation control mode, the direction of the links are controlled with respect to the relative configuration frame defined by Q at (17). Therefore, the direction of the links changes according to the direction of the desired force F_d, which asymptotically converges to $m_y(\ddot{y}_d - ge_3)$ from (16). Furthermore, the magnitude of the virtual control input, or tension, becomes identical for all quadrotors, i.e., $\|\mu_i\| \to \frac{\|F_d\|}{\|\sum_{i=1}^n r_{i_d}\|}$ as $t \to \infty$ for all $i \in \mathcal{I}$.

For the inertial formation control mode, the direction of the links is arbitrarily controlled by q_{i_d} with respect to the inertial frame. In contrast to the relative formation control mode, the sum of the virtual control input is equal to the desired force always at (22), even if $q_i \neq q_{i_d}$. Therefore, this mode exhibits better initial transient responses.

Remark 2: The proposed control system with $n = 1$ can be used for an inverted spherical pendulum on a quadrotor. Since the desired relative configuration satisfying (10), (11) is given by $r_{1_d} = e_3$ for $n = 1$, the desired direction of the first link becomes $q_{1_d} = -\frac{F_d}{\|F_d\|} = -\frac{\ddot{y}_d - ge_3}{\|\ddot{y}_d - ge_3\|}$ at steady-state. As $\|\ddot{y}_d\| < g$ in general, we have $q_{1_d} \cdot e_3 > 0$, i.e., the direction from the quadrotor to the point mass is pointing downward. This is desirable if the quadrotor and the point mass are actually connected by a flexible string, as the corresponding tension along the string becomes positive.

But, if it is assumed that the link is rigid, the sign of the desired direction can be simply reversed to obtain $q_{1_d} = +\frac{F_d}{\|F_d\|}$. The expressions for other parts of the control system remain unchanged. In this case, there is no effect on the steady-state tracking performance as (20) still holds. But, we have $q_{1_d} \cdot e_3 < 0$, which implies that the point mass is above the quadrotor. This corresponds to tracking control of an inverted spherical pendulum attached to a quadrotor.

IV. Numerical Examples

Properties of $n = 4$ quadrotors and a load are given by

$$m_i = 0.755\,\text{kg}, \quad J_i = \text{diag}[0.0820,\ 0.0845,\ 0.1377]\,\text{kgm}^2,$$
$$m_y = 0.4\,\text{kg}, \quad l = [0.6,\ 0.8,\ 0.6,\ 1.0]\,\text{m}.$$

The initial conditions are chosen as

$$y(0) = [1,\ 0,\ 0]^T\,\text{m}, \quad \theta_i = (i-1)90^\circ,\ \phi = 80^\circ,$$
$$q_i(0) = [\cos\theta_i \sin\psi,\ \sin\theta_i \sin\phi,\ \cos\phi]^T, \quad R_i = I.$$

All of the initial velocities are chosen to be zero. The desired trajectory of the load is

$$y_d(t) = 0.8[\sin 2t,\ 0.1t,\ 0.5\cos t]^T\,\text{m}$$

where t is written in seconds. The following three cases are considered.

Relative Formation Control $(n = 4)$*:* The desired relative configuration is chosen as

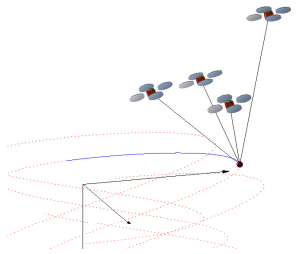
$$r_{i_d} = [\cos\theta_{i_d} \sin\phi_d,\ \sin\theta_{i_d} \sin\phi_d,\ \cos\phi_d]^T,$$
$$\theta_{i_d} = (i-1)90^\circ,\ \phi_d = 30^\circ, \quad s_d = -e_2,\ b_{1_i} = -e_1.$$

Simulation results are illustrated at Fig. 2. The first subfigure is a snapshot of quadrotors, where the dotted red line is the desired trajectory and the solid blue line is the controlled trajectory over the last 1.2 seconds. At the given instant, the desired force F_d is pointing upper left side as the load is moving left, and quadrotors are aligned relative to the direction of F_d in this relative formation control mode. As a result, the overall formation of quadrotors are tilted left. The next subfigures show the position of the load, the attitude errors, and the link direction errors, and these illustrate good tracking performances even for large initial attitude errors that are close to 180°.

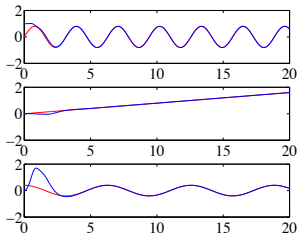
Inertial Formation Control $(n = 4)$*:* The desired inertial configuration of quadrotors is chosen as

$$q_{i_d} = [\cos\theta_{i_d} \sin\phi_d,\ \sin\theta_{i_d} \sin\phi_d,\ \cos\phi_d]^T,$$
$$\theta_{i_d} = (i-2)90^\circ,\ \phi_d = 30^\circ, \quad b_{1_i} = -e_1.$$

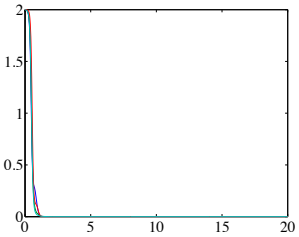
Simulation results are illustrated at Fig. 3. Compared with the previous relative formation control mode, the averaged direction of the links from quadrotors to the mass is pointing downward at Fig. 3(a), as specified by q_{i_d}. Instead, the attitudes of quadrotors are rotated left to generate the desired force F_d pointing upper left side. At Fig. 3(b), it exhibits a better initial transient response than the above relative formation control mode, especially for the third component of y corresponding to height. This is because (22) is satisfied for any q_i, even before q_i asymptotically converges to q_{i_d}.



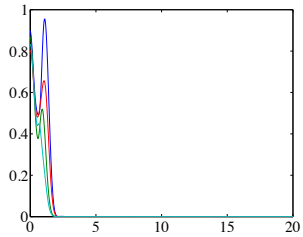
(a) Snapshot at $t = 10.2\,\text{sec}$



(b) Load position (y_d:red, y:blue)

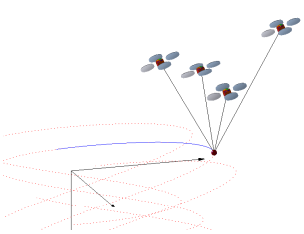


(c) Attitude error of quadrotors $\Psi_{R_i} = \frac{1}{2}\text{tr}\big[I - R_{c_i}^T R_i\big]$

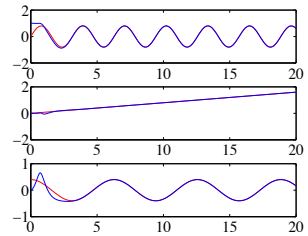


(d) Direction error of links $\Psi_{q_i} = 1 - q_i \cdot q_{i_d}$

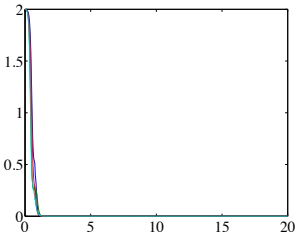
Fig. 2. Relative formation control $n = 4$



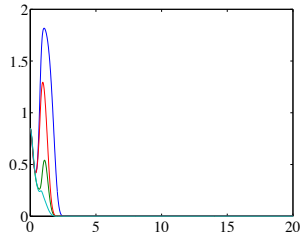
(a) Snapshot at $t = 10.2\,\text{sec}$



(b) Load position (y_d:red, y:blue)



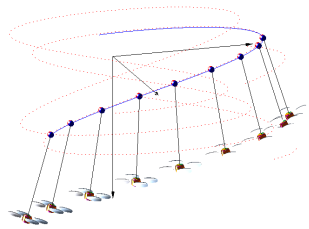
(c) Attitude error of quadrotors $\Psi_{R_i} = \frac{1}{2}\text{tr}\big[I - R_{c_i}^T R_i\big]$



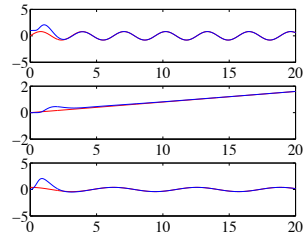
(d) Direction error of links $\Psi_{q_i} = 1 - q_i \cdot q_{i_d}$

Fig. 3. Inertial formation control $n = 4$

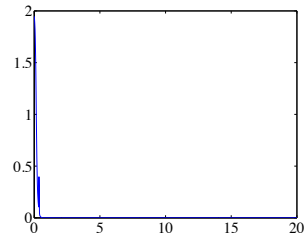
Flying inverted spherical pendulum ($n = 1$): It is assumed that only the first quadrotor is available, and the sign of q_{1_d} is flipped to represent a flying inverted spherical pendulum model as discussed at Remark 2. Fig. 4 illustrates good tracking performances even for large initial errors in the attitude of quadrotor and the direction of the link.



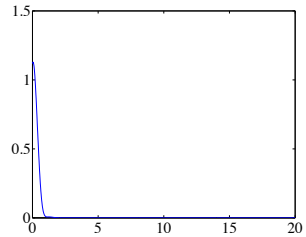
(a) Snapshots at $10.2 \leq t \leq 11.6\,\text{sec}$



(b) Load position (y_d:red, y:blue)



(c) Attitude error of quadrotors $\Psi_{R_i} = \frac{1}{2}\text{tr}\big[I - R_{c_i}^T R_i\big]$



(d) Direction error of links $\Psi_{q_i} = 1 - q_i \cdot q_{i_d}$

Fig. 4. Flying inverted spherical pendulum $n = 1$

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