Nonsmooth Control Barrier Functions for Obstacle Avoidance between Convex Regions

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***Abstract*—In this paper, we focus on non-conservative obstacle avoidance between robots with control affine dynamics with strictly convex and polytopic shapes. The core challenge for this obstacle avoidance problem is that the minimum distance between strictly convex regions or polytopes is generally implicit and non-smooth, such that distance constraints cannot be enforced directly in the optimization problem. To handle this challenge, we employ non-smooth control barrier functions to reformulate the avoidance problem in the dual space, with the positivity of the minimum distance between robots equivalently expressed using a quadratic program. Our approach is proven to guarantee system safety. We theoretically analyze the smoothness properties of the minimum distance quadratic program and its KKT conditions. We validate our approach by demonstrating computationally-efficient obstacle avoidance for multi-agent robotic systems with strictly convex and polytopic shapes. To our best knowledge, this is the first time a real-time QP problem can be formulated for general non-conservative avoidance between strictly convex shapes and polytopes.**

I. Introduction

A. Motivation

Optimization-based approaches usually consider the control and planning algorithms as an optimization problem with obstacle avoidance constraints, where the minimum distance between the robots and the obstacles is enforced to always be positive along the trajectory. The minimum distance functions for these constraints are smooth and differentiable when the shapes of the robots and the obstacles are considered lines, planes, circles, ellipses, hyper-ellipses, etc. However, the smoothness and the differentiability are no longer valid when the robots or the obstacles are considered as general convex regions, such as polytopes [1]. To handle this issue, polytopic obstacle avoidance can be reformulated into smooth constraints with dual variables within model predictive control for discrete-time systems. However, the optimization is computationally heavy for real-time implementation as the constraints in the dual space are still nonlinear and non-convex [2]. Recent progress in the field of control barrier functions permits us to formulate quadratic optimization problems for obstacle avoidance for continuous-time systems [3], although only for differentiable distance functions. The challenges of computational complexity and non-smooth distances between convex regions motivate us to describe the minimum distance constraint between convex regions in a differentiable and smooth manner. We will show that this can be achieved for polytopic and strictly convex sets by incorporating KKT conditions into the control barrier function constraints. The proposed formulations allow us to reformulate a set of non-differentiable and non-convex obstacle avoidance constraints into convex and differentiable ones through nonsmooth control barrier functions [4] and achieve real-time computation with convex quadratic programs. The resulting control policy allows for tight obstacle avoidance maneuvers, as shown in Fig. 1.

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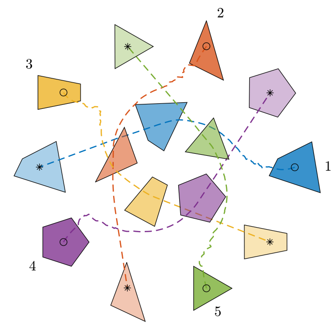
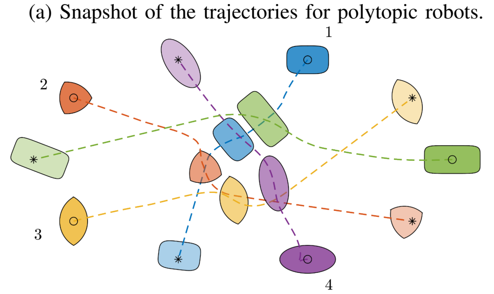


(a) Snapshot of the trajectories for polytopic robots.



(b) Snapshot of the trajectories for strictly convex-shaped robots.

Fig. 1: Snapshots of the trajectories for polytopic and strictly convex-shaped robots using our proposed formulation. The 'o' markers indicate the initial positions, while '*' indicates the final position. The shape of each robot is different, with three having unicycle dynamics and the others having integrator dynamics, which illustrates the versatility of our formulation. The robot shapes are not over-approximated, allowing for tight maneuvers while guaranteeing safety.

B. Related Work

Collision-free maneuvering for autonomous systems in an obstacle-dense environment is a challenging problem. In almost all existing cybernetics systems, one core behavior is required: to avoid collision with surrounding obstacles.

1) Obstacle Avoidance in Planning and Control: In the last two decades, a majority of obstacle avoidance problems have been considered using optimization-based trajectory planning algorithms through dynamic programming-related approaches [5], such as trajectory parametrization [6], [7], convex optimizations [8], [9], sequential optimization [10]–[12], LQR/LQG [13]–[15], and model predictive control [16]–[18]. All the above approaches and their variants are related to optimization techniques that use the gradient of the discrete-time system dynamics. Recently, obstacle avoidance has also been achieved through control approaches for continuous-time systems, such as with barrier-based approaches [4], [19]–[21].

2) Necessity of Convex Obstacle Avoidance: To achieve obstacle avoidance in planning and control problems, the minimum distance function between the robots and the obstacles is considered [22]. This will generate repulsion from obstacles, where either additional terms are added in the cost function or inequalities are enforced in the constraints for the optimization problems. The additional cost terms and constraints are functions of the minimum distance, which are required to be smooth and differentiable. Therefore, the shapes of the robots and the obstacles are often approximated as lines, planes, circles, ellipses, and spheres [12], [15], [22], [23]. These approximations could result in a conservative obstacle avoidance behavior and deadlock for trajectory generation problems or navigation tasks.

3) Existing Work on Polytopic Obstacle Avoidance: The existing optimization-based approaches for polytopic obstacle avoidance can be classified as mixed-integer programming and model predictive control (MPC) with duality. Mixed-integer programming [24]–[26] was initially developed for hybrid systems and then generalized for polytopic avoidance [27], [28] since the nearest pair of points for minimum distance between polytopes are on different edges, which could be described as mixed-integer variables. For real-time computation, mixed-integer programming can only be applied to linear systems. Dual analysis was introduced into the MPC problem [2] and allows us to define obstacle avoidance constraints in trajectory generation problems. This approach is computationally faster than mixed-integer programming but still requires non-convex optimization. Therefore, this approach can only be used online for linear systems [29] and in offline planning for nonlinear systems [30]. However, the optimization problems are always non-convex for nonlinear systems for mixed-integer programming and model predictive control with duality, meaning real-time computation cannot be guaranteed for general nonlinear systems. Approximating polytopes into circles or spheres can make the problem convex, but the approximation is conservative and could generate deadlock. This motivates us to seek convex optimizations for non-conservative polytopic avoidance.

4) Obstacle Avoidance between Strictly Convex Regions: As discussed above, one core challenge for obstacle avoidance between polytopes is the non-smoothness and the implicit expression of the minimum distance. The distance function between strictly convex sets is also non-smooth and implicit in general. Only a few existing works [31], [32] attempt to solve obstacle avoidance between strictly convex regions. Moreover, they all assume specific forms of convex regions, e.g., ellipses or spheres. This motivates us to propose a general way to solve obstacle avoidance between strictly convex regions.

5) Convex Optimizations with Control Barrier Functions: Recent research in control barrier functions (CBFs) brings the possibility of solving obstacle avoidance with convex optimizations in the scope of control problems. The studies on control barrier functions have used a variety of approaches that can be best described as "Lyapunov-like". In other words, these functions yield invariant super-level sets such that one can guarantee safety if these level sets are contained within the safe set [33]. CBF-QP [34] uses quadratic programs to find the minimum perturbation of a given feedback controller to guarantee safety. Control Lyapunov functions (CLFs) [35], [36] can be applied to stabilize the closed-loop dynamics of both linear and nonlinear dynamical systems [37]. Together with CBFs, the framework of CLF-CBF-QP [3] incorporates control barrier functions with control Lyapunov function-based quadratic programs, which enables handling safety-critical constraints effectively in real-time with low complexity of solving QPs. These safety-critical constraints with control barrier functions are usually used for obstacle avoidance. To achieve obstacle avoidance, these control barrier functions are a function of the distance between different shapes of robots and obstacles, including point-mass [38], circles [39], ellipses [40], parabolas [41] and high-dimensional spheres [42]. All these existing work motivates us to apply control barrier functions to deal with obstacle avoidance for convex regions. However, the nominal CBF-QP and CLF-CBF-QP formulations, and their variants, require the control barrier functions to be differentiable, while the distance functions between convex regions are not only non-smooth but do not even hold explicit forms [43], [44]. To overcome this challenge, in this paper, we seek differentiable control barrier functions with convex obstacle avoidance in the dual space instead of the primal space in the optimization problem.

C. Contributions

Preliminary results of this work were published in [1], which focused on obstacle avoidance between polytopes. We further elaborate on the results of this work and also provide a method for obstacle avoidance between strictly convex regions. The contributions of this paper are as follows:

- We provide a novel optimization algorithm for collision avoidance between a large class of robots whose geometries are described by either a polytope or a strictly convex set and whose dynamics are characterized by a nonlinear control affine system. The proposed method is also not conservative, meaning the algorithm allows for tight avoidance maneuvers in obstacle-filled environments.
- We use the results from nonsmooth analysis to show the differentiability properties of the KKT solution of the minimum distance optimization problem. Nonsmooth analysis is also used to prove the overall safety properties of the system under our proposed controller.
- We formulate the problem in continuous-time, and the control optimization problem is a QP and thus can be solved in real-time.

D. Notation

For $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, 2, ..., n\}$. Superscripts of variables, such as x^i, denote the robot index. Subscripts of variables, such as A_k, denote the row index of vectors or matrices, and stylized subscripts, such as $A_{\mathcal{R}}$, denote the submatrix obtained by selecting those rows whose index lies in the set $\mathcal{R}$. For mathematical proofs, subscripts of variables with parenthesis, such as $a_{(m)}$, denote the m-th element in the sequence $\{a_{(m)}\}$. Tab. I summarizes the symbols used in the paper.

TABLE I: List of Symbols and Notations

Symbol	Meaning
$x^i \in \mathcal{X}^i \subseteq \mathbb{R}^n$	System state of Robot i
$s^*(x) \in \mathbb{R}^l$	Optimal separating vector between $\mathcal{C}^i(x^i)$ and $\mathcal{C}^j(x^j)$; minimizer of (9)
$u^i \in \mathcal{U}^i \subseteq \mathbb{R}^m$	System input for Robot i
$A^i(x^i, z^i) \in \mathbb{R}^{r^i}$	Constraints defining $\mathcal{C}^i(x^i)$ for the strictly convex set case; (10)
$f^i(x^i)$	Drift vector field for Robot i; (1)
$\mathcal{J}^i(x^i, z^i)$	Set of active constraints at z^i; (11)
$g^i(x^i)$	Input vector field for Robot i; (1)
$\lambda^i \in \mathbb{R}^{1\times r^i}$	Dual variables for Robot i
$F[\cdot]$	Filippov operator; (4)
$L(x, z, \lambda)$	Lagrangian function; (13)
$\mathcal{C}^i(x^i) \subseteq \mathbb{R}^l$	Geometric region occupied by Robot i
$\mathcal{J}^i_k(x^i)$	Set of active and inactive constraints at an optimal solution z^{*i}, $k = \{0, 1, 2\}$; (20), (22), (29)
$h(x) \in \mathbb{R}_{\geq 0}$	Square of the minimum distance between $\mathcal{C}^i(x^i)$ and $\mathcal{C}^j(x^j)$; (12), (25)
$A^i(x^i) \in \mathbb{R}^{r^i \times l}$, $b^i(x^i) \in \mathbb{R}^{r^i}$	Half-space constraints defining $\mathcal{C}^i(x^i)$ for the polytope case; (24)
z^i	Any point in Robot i, $z^i \in \mathcal{C}^i(x^i)$
$\mathrm{Aff}^i(x^i)$	Parallel affine space defining x^i; (30)
$\mathcal{S} \subseteq \mathcal{X}^i \times \mathcal{X}^j$	Set of safe states; (6)
$\Lambda(x, \lambda)$	Lagrangian dual function; (27)

E. Paper Structure

The paper is organized as follows. A brief introduction to discontinuous dynamical systems and nonsmooth control barrier functions is presented in Sec. II. In Sec. III, we analyze the general smoothness properties of the optimal solutions of the minimum distance function, including uniqueness, continuity, and differentiability. We analyze the continuity and differentiability properties of the minimum distance problem between strictly convex sets in Sec. IV and propose an obstacle avoidance formulation with NCBFs for strictly convex sets in Sec. V. The properties of dual solutions for polytopic sets are discussed in Sec. VI, and the corresponding optimization formulation with NCBFs for polytopes is then presented in Sec. VII. In Sec VIII, we numerically validate our approach for obstacle avoidance with NCBFs, and concluding remarks are presented in Sec. IX.

II. Background

In this paper, we consider the enforcement of safety constraints between multiple controllable robots and obstacles. Each controlled robot is associated with some states, control inputs, system dynamics which describe the state evolution, and geometries which describe the physical domain occupied by each of the robot links. Enforcing safety for such robots requires their control policies to guarantee that no two robots or obstacles collide with each other, i.e. the minimum distance between any two robots is always greater than zero.

Consider N controlled robots with the i-th robot having states $x^i \in \mathcal{X}^i \subset \mathbb{R}^n$ and nonlinear, control-affine dynamics:

$$\dot{x}^i(t) = f^i(x^i(t)) + g^i(x^i(t))u^i(t), \quad i \in [N], \tag{1}$$

where $f^i : \mathcal{X}^i \to \mathbb{R}^n$ and $g^i : \mathcal{X}^i \to \mathbb{R}^{n\times m}$ are continuous functions, $u^i(t) \in \mathcal{U}^i \subset \mathbb{R}^m$, and $[N] = \{1, 2, ..., N\}$. We assume $\mathcal{X}^i$ to be an open connected set and $\mathcal{U}^i$ a convex compact set.

ODEs of the form of (1), where the RHS is a continuous function, are guaranteed to have a solution, but the solution might be non-unique. Although restricting the functions f^i, g^i, and u^i in (1) to be Lipschitz continuous functions would guarantee uniqueness of solutions, imposing Lipschitz continuity on u^i is too restrictive. In this work, we assume that the geometry of any robot can be described by a polytope, see Fig. 4, or a strictly convex set, see Fig. 3, which have nonsmooth boundaries, so the minimum distance between any pair of such geometries is not differentiable. This means that the control input obtained using the distance function might not be continuous, as is noted in the subsequent sections. Therefore, we first present some background about solutions of ODEs with a discontinuous RHS.

A. Discontinuous Dynamical Systems

To have a well-defined notion of a solution to an ODE with a discontinuous RHS, we can study the properties of differential inclusions of the form:

$$\dot{x}^i(t) \in F^i(x^i(t)), \quad x^i(0) = x^i_0, \tag{2}$$

where $F^i : \mathcal{X}^i \to 2^{\mathbb{R}^n}$ is a set-valued map. Here $2^{\mathbb{R}^n}$ denotes the power set of $\mathbb{R}^n$. We also need the notion of continuity of set-valued maps.

Definition 1. *(Semi-continuity) [45, Sidebar 7] A set-valued map $\Gamma : \mathcal{X} \to 2^{\mathbb{R}^n}$ is said to be upper semi-continuous (respectively, lower semi-continuous) at $a \in \mathcal{X}$, if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall x \in \mathcal{B}_\delta(a)$, $\Gamma(x) \subset \Gamma(a) + \mathcal{B}_\epsilon(0)$ (respectively, $\Gamma(a) \subset \Gamma(x) + \mathcal{B}_\epsilon(0)$). Here $\mathcal{B}_r(x) := \{y : \|x - y\| < r\}$ is an open ball of radius r around x, and the addition between sets is defined as Minkowski addition. For sets A and B, the Minkowski addition is defined as $A + B := \{a + b : a \in A, b \in B\}$. Γ is continuous at $a \in \mathcal{X}$ if it is both upper and lower semi-continuous at $a \in \mathcal{X}$.*

For a differential inclusion of the form of (2), a notion of a solution can be defined as follows.

Definition 2. *(Caratheodory Solution) [45] A Caratheodory solution on $[0, T]$ to (2) is an absolutely continuous map $x^i : [0, T] \to \mathcal{X}^i$ which satisfies (2) for almost all $t \in [0, T]$, i.e. the set of points where (2) is not satisfied has measure zero.*

We now examine the dynamical system (1) with a discontinuous feedback control input $u(x)$ and convert it into the form (2) to obtain a Caratheodory solution for the system. To do this, we use the Filippov operator. For a vector field $f^i : \mathcal{X}^i \to \mathbb{R}^n$, the Filippov operator on f^i, $F[f^i] : \mathcal{X}^i \to 2^{\mathbb{R}^n}$, is defined as [45, Eq. (30)],

$$F[f^i](x) := \mathrm{co}\{\lim_{k\to\infty} f^i(x_{(k)}) : x_{(k)} \to x, x_{(k)} \notin \mathcal{Q}_d, \mathcal{Q}\}, \tag{3}$$

where 'co' stands for convex hull, $\mathcal{Q}_d$ is the zero-measure set of the points of discontinuities of f^i, and $\mathcal{Q}$ is an arbitrary zero-measure set. Note that $F[f^i]$ is a set-valued map.

Let $u^i : \mathcal{X}^i \to \mathcal{U}^i$ be some measurable feedback control law. The discontinuous dynamical system (1) can be converted to a differential inclusion of the form (2), using the Filippov operator $F[\cdot]$ as :

$$\dot{x}^i(t) \in F[f^i + g^i u^i](x^i(t)) \tag{4}$$

We can now define a solution of (1) as a solution of (4).

Definition 3. *(Filippov Solution) [45, Eq. (21)] A Filippov solution of (1) is a Caratheodory solution of (4).*

Caratheodory solutions are defined for differential inclusions of the form (2), whereas Filippov solutions are defined for ODEs of the form (1) by converting the ODE into a differential inclusion using (4). The following result guarantees the existence of Filippov solutions.

Proposition 1. *(Existence of Filippov Solutions) [45, Prop. 3] The map $F[f^i + g^i u^i] : \mathcal{X}^i \to 2^{\mathbb{R}^n}$ is upper semi-continuous and is non-empty, convex, and compact at each $x^i \in \mathcal{X}^i$. Then, for all $x^i_0 \in \mathcal{X}^i$,* (4) *has a Caratheodory solution with $x^i(0) = x^i_0$, which is a Filippov solution to* (1)*.*

We also make a note of the following properties of the Filippov operator:

Property 1. *(Distributive Property) By the sum and product rules for Filippov operators [45, Eq. (26-27)] and the continuity of f^i and g^i,* (4) *is equivalent to*

$$\dot{x}^i(t) \in f^i(x^i(t)) + g^i(x^i(t))F[u^i](x^i(t)). \tag{5}$$

Property 2. *Let $\Gamma : \mathcal{X} \to \mathcal{U}$, $\mathcal{U} \subseteq \mathbb{R}^m$, be a map such that for all x, $\Gamma(x) \in \mathcal{F}_u(x)$, where $\mathcal{F}_u : \mathcal{X} \to 2^{\mathbb{R}^m}$ is a pointwise non-empty, convex, and compact set-valued map. Also let Γ be such that for all x and sequences $\{x_{(m)}\} \to x$, $\lim_{m\to\infty} \Gamma(x_{(m)})=u \Rightarrow u \in \mathcal{F}_u(x)$. Then, by the definition of Filippov operator* (4)*, $F[\Gamma](x) \subseteq \mathcal{F}_u(x)$, $\forall x \in \mathcal{X}$.*

B. Nonsmooth Control Barrier Functions

To enforce obstacle avoidance between Robots i and j, we want the minimum distance between them to be greater than 0. Let $h^{ij}(x^i, x^j)$ be the minimum distance between the geometries of Robot i at state x^i and Robot j at state x^j. Note that the minimum distance between two non-empty sets is always well-defined and non-negative. We can define safe states as the states (x^i, x^j) where $h^{ij}(x^i, x^j)$ is greater than zero [33], as

$$\mathcal{S}^{ij} := \{(x^i, x^j) : h^{ij}(x^i, x^j) > 0\}^{cl}, \tag{6}$$

where $(\cdot)^{cl}$ denotes closure of a set.

Remark 1. *We define $\mathcal{S}^{ij}$ as the closure of the actual safe set to ensure that $\mathcal{S}^{ij}$ is a closed set, but this may also introduce states that are not safe into $\mathcal{S}^{ij}$. Under certain regularity assumptions on the geometries of the robots and obstacles, taking the closure of the safe set is not detrimental to the obstacle avoidance problem. Taking the closure of the safe set introduces only those unsafe states in which the obstacle and the robot have an intersection of measure zero [46].*

Let, for all Robots i, $u^i : \mathcal{X}^i \to \mathcal{U}^i$ be a measurable feedback control law, with the corresponding Filippov solution as $x^i(t)$ for $t \in [0, T]$. We consider the system to be safe if for any two Robots i and j, $(x^i(t), x^j(t)) \in \mathcal{S}^{ij}$ $\forall t \in [0, T]$. We want to enforce conditions on u^i to guarantee the system's safety. However, simply enforcing the condition that $h^{ij}(x^i(t), x^j(t)) > 0$ at a particular time is not useful since h^{ij} does not explicitly depend on the control input. The control input influences the state evolution, so we can enforce a safety condition on the time-derivative $\dot{h}^{ij}$. To achieve this, we first adapt the definition of barrier functions from [4, Def. 4].

Definition 4. *[4, Def. 4] (Nonsmooth Control Barrier Function) A locally Lipschitz continuous function $h : \mathcal{X} \to \mathbb{R}$ is a nonsmooth control barrier function (NCBF) if there exists a measurable control law $u : \mathcal{X} \to \mathcal{U}$ such that the set $\mathcal{S} = \{x \in \mathcal{X} : h(x) \geq 0\}$, is forward invariant for the closed-loop system.*

If h is a locally Lipschitz continuous function, it is also absolutely continuous [45]. By Def. 3, the Filippov solution $x(t)$ is absolutely continuous, and so $h \circ x$ is also absolutely continuous and is differentiable almost everywhere [45]. Then the following lemma can be used to guarantee safety:

Lemma 1. *[4, Lem. 2] Let $\alpha > 0$ be a constant, and $h : [0, T] \to \mathbb{R}$ be an absolutely continuous function. If*

$$\dot{h}(t) \geq -\alpha \cdot h(t) \tag{7}$$

for almost all $t \in [0, T]$ and $h(0) > 0$, then $h(t) \geq h(0)e^{-\alpha t} > 0$ $\forall t \in [0, T]$.

NCBFs are a generalization of Control Barrier Functions (CBFs) [33] to nonsmooth functions, and the constraint (7) is called the NCBF constraint. The safety for the system (1) can be enforced by choosing h^{ij} as an NCBF, but, as opposed to previous work on CBFs, the minimum distance h^{ij} is implicitly calculated as the solution of a minimization problem. In the following sections, we will show how to explicitly enforce (7) for the minimum distance function h^{ij} for both strictly convex sets and polytopic sets. We first analyze the general properties of the minimum distance function for convex sets in Sec. III. In Sec. IV and Sec. VI, we derive a set of smooth, explicit constraints for the derivative of the minimum distance for strictly convex sets and polytopes respectively. These constraints are then used to enforce the NCBF constraint (7) and prove safety in Sec. V and Sec. VII. A flowchart of results is shown in Fig. 2.

III. Uniqueness / Continuity / Differentiability for the Minimum Distance Problem

This section identifies general results on the smoothness of the minimum distance between a pair of convex sets. The first two subsections discuss the uniqueness and continuity of optimal solutions, and the third subsection focuses on the optimal value's differentiability. We make incremental assumptions on the convex sets to determine the corresponding properties in each subsection. The results in the section justify that the minimum distance function can be used as a candidate NCBF and is the base for the results in the subsequent sections.

For simplicity, we will restrict our discussion to a pair of Robots i and j and generalize when necessary. We also define $\mathcal{X} := \mathcal{X}^i \times \mathcal{X}^j$, $\mathcal{U} := \mathcal{U}^i \times \mathcal{U}^j$, $x := (x^i, x^j) \in \mathcal{X}$, $h(x) := h^{ij}(x^i, x^j)$, $\mathcal{S} := \mathcal{S}^{ij}$, and $u := (u^i, u^j) \in \mathcal{U}$ for the Robots i and j.

A. Uniqueness of Optimal Solutions

Let $\mathcal{C}^i : \mathcal{X}^i \to 2^{\mathbb{R}^l}$ and $\mathcal{C}^j : \mathcal{X}^j \to 2^{\mathbb{R}^l}$ be set-valued maps defined from the states to the l-dimensional physical space. $\mathcal{C}^i(x^i), \mathcal{C}^j(x^j) \subset \mathbb{R}^l$ represent the physical space occupied by the Robots i and j at states x^i and x^j respectively.

Assumption 1. *We assume the following for $\mathcal{C}^i$, for all i:*

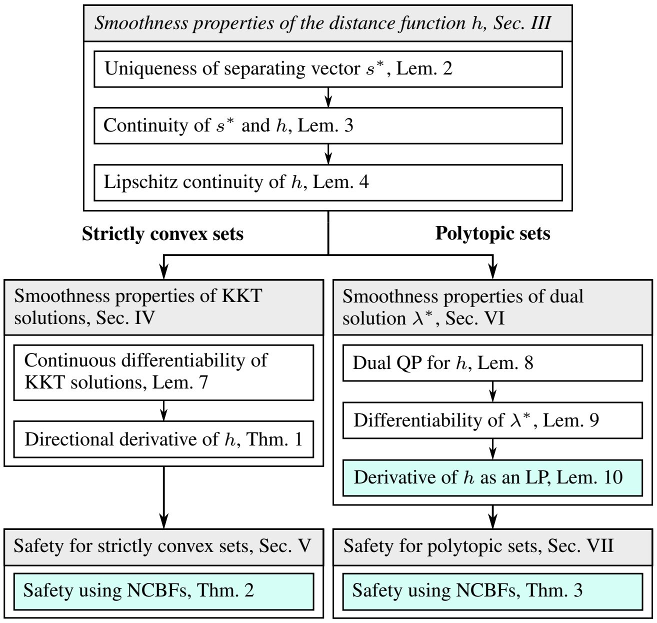


Fig. 2: A flowchart of the important results in the paper. The results in blue are the main contributions of the paper. Sec. III identifies smoothness properties of the minimum distance optimization for general convex sets. Sec. IV and Sec. V cover smoothness and safety for strictly convex sets, respectively. Sec. VI and Sec. VII cover smoothness and safety for polytopes and are independent of the sections on strictly convex sets.

(a) $\mathcal{C}^i(x^i)$ is convex, compact, and has a non-empty interior for all $x^i \in \mathcal{X}^i$.

(b) The set-valued maps $\mathcal{C}^i$ are continuous, i.e. both lower and upper semi-continuous.

These are natural assumptions since we expect the shape of the robot to be bounded, have non-zero volume, and change continuously with the robot's state.

The minimum distance $h(x)$ between $\mathcal{C}^i(x^i)$ and $\mathcal{C}^j(x^j)$ is defined by the optimization problem,

$$\begin{aligned} h(x) = \min_{z} \quad & \|z^i - z^j\|_2^2 \\ \text{s.t.} \quad & z^i \in \mathcal{C}^i(x^i), z^j \in \mathcal{C}^j(x^j). \end{aligned} \tag{8}$$

Variables $z^i, z^j \in \mathbb{R}^l$ denote points inside the sets $\mathcal{C}^i(x^i)$ and $\mathcal{C}^j(x^j)$ respectively, and let $z := [z^{iT}, z^{jT}]^T$. Any value of z that is a minimizer of (8) is referred to as an optimal solution.

A separating vector is defined as a vector from $\mathcal{C}^j(x^j)$ to $\mathcal{C}^i(x^i)$, which has the minimum length. The separating vector s^* is obtained as a solution to the optimization problem,

$$\begin{aligned} h(x) = \min_{s} \quad & \|s\|_2^2 \\ \text{s.t.} \quad & s \in \mathcal{C}^i(x^i) - \mathcal{C}^j(x^j), \end{aligned} \tag{9}$$

where $\mathcal{C}^i(x^i) - \mathcal{C}^j(x^j) := \{s = z^i - z^j : z^i \in \mathcal{C}^i(x^i), z^j \in \mathcal{C}^j(x^j)\}$ is the Minkowski addition of $\mathcal{C}^i(x^i)$ and $-\mathcal{C}^j(x^j)$. A non-zero separating vector implies that the two sets are disjoint. The separating vector is also unique, as shown in the following lemma.

Lemma 2. *(Uniqueness of Separating Vector) If Assumption 1.a holds, then at least one optimal solution exists, and the separating vector s^* defined as $s^* = z^{*i} - z^{*j}$ is unique for all pairs of optimal solutions $z^* := (z^{*i}, z^{*j})$ of* (8)*.*

Further, if $\mathcal{C}^i(x^i)$ is strictly convex and $\mathcal{C}^i(x^i) \cap \mathcal{C}^j(x^j) = \emptyset$, then both z^{*i} and z^{*j} are unique.

Proof. The proof is provided in Appendix A. □

For any $x \in \mathcal{X}$, $s^*(x)$ is defined as the unique minimizer of (9). The optimal solution of (8), if unique, is defined as $z^*(x) = (z^{*i}(x), z^{*j}(x))$. To show the continuity of h and s^*, we consider the additional assumption on the continuity of $\mathcal{C}^i$, Assumption 1.b.

B. Continuity of Optimal Solutions

The following result shows that, under Assumption 1, the continuity of the minimum distance h and the separating vector s^* can be established using (9).

Lemma 3. *(Continuity of Separating Vector) If Assumption 1 holds, then the minimum distance function h and the separating vector s^* are continuous for all $x \in \mathcal{X}$.*

Further, if $\mathcal{C}^i(x^i)$ is strictly convex for all $x^i \in \mathcal{X}^i$ and $\mathcal{C}^i(x^i) \cap \mathcal{C}^j(x^j) = \emptyset$ at x, then the unique minimizer $z^*(x)$ of (8) *is continuous at x.*

Proof. The proof is provided in Appendix B. □

Lem. 3 implies that the separating vector, which is the normal vector of the optimal hyperplane separating $\mathcal{C}^i$ and $\mathcal{C}^j$, moves continuously with the states of the robots. To derive the differentiability properties for general convex sets, we need to impose further restrictions on the structure of the convex sets.

C. Differentiability of Optimal Solutions

Differentiability of the minimum distance function h requires stronger assumptions on the convex sets $\mathcal{C}^i(x^i)$ and $\mathcal{C}^j(x^j)$ than for continuity. Let $\mathcal{C}^i$ be of the form,

$$\mathcal{C}^i(x^i) = \{z^i \in \mathbb{R}^l : A^i_k(x^i, z^i) \leq 0, k = 1, ..., r^i\}, \tag{10}$$

where $A^i_k : \mathcal{X}^i \times \mathbb{R}^l \to \mathbb{R}$, and r^i is the number of constraints used to define $\mathcal{C}^i(x^i)$.

Assumption 2. *We assume the following for all i and k:*

(a) $A^i_k(x^i, \cdot)$ is convex in $z \in \mathbb{R}^l$, $\forall x^i \in \mathcal{X}^i$.

(b) $\mathcal{C}^i(x^i)$ is a compact set with a non-empty interior. Moreover, $\exists z \in \mathcal{C}^i(x^i)$ such that $A^i(x^i, z) < 0$.

(c) A^i_k is continuous on $\mathcal{X}^i \times \mathbb{R}^l$. Further, for any $x^i \in \mathcal{X}^i$, there is an open convex set $\mathcal{W} \supset \mathcal{C}^i(x^i)$ and a neighborhood $\mathcal{N}$ of x^i such that for each $z \in \mathcal{W}$, $A^i_k(\cdot, z)$ is Lipschitz continuous in $\mathcal{N}$ with a Lipschitz constant independent of z, i.e., $\forall y^1, y^2 \in \mathcal{N}, z \in \mathcal{W}$

$$|A^i_k(y^1, z) - A^i_k(y^2, z)| \leq L_x \|y^1 - y^2\|.$$

$\mathcal{C}^i(x^i)$ is the intersection of the 0-sublevel sets of $A^i_k(x^i, \cdot)$. So, if Assumption 2.a holds, $\mathcal{C}^i(x^i)$ is convex for all $x^i \in \mathcal{X}^i$. Thus, if Assumptions 2.a and 2.b hold, $\mathcal{C}^i(x^i)$ is a convex, compact set with a non-empty interior, and Lem. 3 holds. The Lipschitz property in Assumption 2.c guarantees that $\mathcal{C}^i(x^i)$ changes continuously with the state x^i and, by Lem. 3, that h is continuous. Note, however, that $\mathcal{C}^i(x^i)$ need not be Lipschitz continuous [47, Ex. 1]. Further, h is also locally Lipschitz continuous, as shown in the following lemma:

Lemma 4. *(Lipschitz continuity of minimum distance) If Assumptions 2.b and 2.c hold, then the minimum distance $h(x)$ is locally Lipschitz continuous.*

Proof. The proof is provided in Appendix C. □

Although the minimum distance $h(x)$ is Lipschitz continuous under Assumption 2, the optimal solution of the minimum distance problem (8) (if unique) might be non-Lipschitz. One such example where the optimal solution of a quadratic program, with smooth dependence on parameters, is non-Lipschitz is provided in [48].

This section discussed some general smoothness properties for minimum distance problems. The analysis in this section supports all the theorems and proofs in the rest of the paper. With additional assumptions, the discussion for smoothness properties is then extended to KKT solutions for strictly convex sets and to dual solutions for polytopes, in Sec. IV and Sec. VI respectively. Based on the smoothness properties proved in Sec. IV and Sec. VI, we propose optimization formulations for obstacle avoidance with NCBFs for strictly convex sets and polytopes in Sec.V and Sec. VII respectively. An outline of the results in this section and the following sections is provided in Fig. 2.

IV. Smoothness Properties for Strictly Convex Sets with Dual Analysis

This section discusses continuity and differentiability properties for strictly convex sets in depth. As discussed in the previous section, h is locally Lipschitz, meaning it can be used as a candidate NCBF. However, h is the solution to an optimization problem, and it is unclear how to obtain $\dot{h}$. This section aims to derive a set of explicit constraints to calculate $\dot{h}(x)$, which will then be used to derive the NCBF formulation for strictly convex sets in the next section.

Consider a general class of robots whose geometries are strictly convex and can be explicitly represented by sufficiently smooth functions, as defined in (10). For a given x^i and a point $z^i \in \mathcal{C}^i(x^i)$, define the set of active indices as,

$$\mathcal{J}^i(x^i, z^i) = \{k \in [r^i] : A^i_k(x^i, z^i) = 0\}, \tag{11}$$

i.e. the set of indices of constraints that are active at z^i.

Assumption 3. *We assume the following for all i:*

(a) A^i_k is twice continuously differentiable on $\mathcal{X}^i \times \mathbb{R}^l$ $\forall k$.

(b) $A^i_k(x^i, \cdot)$ is strongly convex in $z \in \mathbb{R}^l$, i.e. $\nabla^2_z A^i_k(x^i, z) \succ 0$, $\forall k$.

(c) The set $\mathcal{C}^i(x^i)$ satisfies linear independence constraint qualification (LICQ), i.e. the set of gradients of active constraints, $\{\nabla_z A^i_k(x^i, z^i) : k \in \mathcal{J}^i(x^i, z^i)\}$ is linearly independent for all $z^i \in \mathcal{C}^i(x^i)$ and $x^i \in \mathcal{X}^i$.

(d) For all $x^i \in \mathcal{X}^i$, $\mathcal{C}^i(x^i)$ is a compact set and has a non-empty interior.

Next, we show how Assumption 3 and the results from Sec. III can be used to determine the smoothness properties of the minimum distance between strictly convex sets. In particular, Assumption 3 implies that Assumption 2 holds.

By Assumption 3.b, the 0-sublevel set of $A^i_k(x^i, \cdot)$ is strictly convex, i.e. if $A^i_k(x^i, z^i_1) = A^i_k(x^i, z^i_2) = 0$, then $A^i_k(x^i, \lambda z^i_1 + (1-\lambda) z^i_2) < 0$, $\forall \lambda \in (0, 1)$. Thus, the set $\{z^i : A^i_k(x^i, z^i) \leq 0\}$ and consequently $\mathcal{C}^i(x^i)$ is strictly convex. By Lem. 2, there exists a unique solution to (8) $\forall x \in \mathcal{S}$, denoted by $z^*(x)$.

Since $A^i : \mathcal{X}^i \times \mathbb{R}^l \to \mathbb{R}^{r^i}$ is a locally Lipschitz vector function (by Assumption 3.a) and $\mathcal{C}^i(x^i)$ is non-empty, the set $\mathcal{C}^i(x^i) = \{z \in \mathbb{R}^l : A^i(x^i, z) \leq 0\}$ is upper and lower semi-continuous. By Lem. 3, the minimum distance function h is continuous and by the strict convexity of $\mathcal{C}^i(x^i)$, $z^*(x)$ is continuous at $x \in \mathcal{S}$.

Let $\mathcal{W} \supset \mathcal{C}^i(x^i)$ be an open, convex, and bounded set. Since A^i_k is continuously differentiable for all k, $A^i_k(\cdot, z)$ is locally Lipschitz continuous for each $z \in \mathbb{R}^l$. Moreover, since $\mathcal{W}$ is bounded, $A^i_k(\cdot, z)$ is Lipschitz continuous around $x^i \in \mathcal{X}^i$ for each $z \in \mathcal{W}$ with a Lipschitz constant independent of $z \in \mathcal{W}$. Thus Assumption 3 ensures that Assumption 2 holds. All conditions for Lem. 4 are satisfied; thus, h is locally Lipschitz continuous. Lipschitz continuity of h means that it can be used as a candidate NCBF.

A. KKT Conditions for the Minimum Distance Problem

Consider the minimum distance optimization problem (8) written as

$$\begin{aligned} h(x) = \min_{z} \quad & \|z^i - z^j\|_2^2 \\ \text{s.t.} \quad & A^i(x^i, z^i) \leq 0, A^j(x^j, z^j) \leq 0. \end{aligned} \tag{12}$$

The primal optimal solution $z^*(x)$ is a global optimum for (12), and we can obtain the first-order necessary conditions that $z^*(x)$ must satisfy using KKT conditions.

For a given $x \in \mathcal{X}$, we define the Lagrangian function $L : \mathcal{X} \times \mathbb{R}^{2l} \times \mathbb{R}^{r^i} \times \mathbb{R}^{r^j} \to \mathbb{R}$ as, [49, Chap. 5]

$$\begin{aligned} L(x^i, x^j, z^i, z^j, \lambda^i, \lambda^j) = \|z^i - z^j\|_2^2 + \lambda^i A^i(x^i, z^i) \\ + \lambda^j A^j(x^j, z^j), \end{aligned} \tag{13}$$

where $\lambda^i \in \mathbb{R}^{1 \times r^i}$ and $\lambda^j \in \mathbb{R}^{1 \times r^j}$ are the dual variables corresponding to the inequality constraints. We denote the dual variables as $\lambda := [\lambda^i, \lambda^j]$ and the Lagrangian function as $L(x, z, \lambda)$. For simplicity of notation, we also define $A(x, z) = [A^{iT}(x^i, z^i), A^{jT}(x^j, z^j)]^T$.

The Karush-Kuhn-Tucker (KKT) conditions are necessary optimality conditions for (12) [49, Chap. 5]. The KKT conditions state that, for each $x \in \mathcal{X}$, there exists KKT solution, (z^*, λ^*), such that the following constraints are satisfied:

$$\nabla_z L(x, z^*, \lambda^*) = 0, \tag{14a}$$

$$\lambda^* A(x, z^*) = 0, \tag{14b}$$

$$\lambda^* \geq 0, \tag{14c}$$

$$A(x, z^*) \leq 0, \tag{14d}$$

where (14a) is the stationarity condition along the set of feasible directions, (14c) is the non-negativity condition for the dual variables, and (14d) is the primal feasibility condition for z^*. The conditions (14b) are called complementary slackness conditions. Note that because of the non-negativity (14c) and primal feasibility (14d) conditions, $\lambda^{*i}_k A^i_k(x^i, z^{*i}) = 0$ $\forall k \in [r^i]$, and similarly for the Robot j. Thus if the constraint $A^i_k(x^i, z^{*i})$ is inactive at z^{*i}, the corresponding dual variable $\lambda^{*i}_k = 0$. However, both $A^i_k(x^i, z^{*i})$ and λ^{*i}_k can be zero simultaneously. Strict complementary slackness condition holds for the KKT solution (z^*, λ^*) if for all $k \in [r^i], \lambda^{*i}_k > 0$ whenever $A^i_k(x^i, z^{*i}) = 0$ (and similarly for j).

Since the minimum distance optimization problem (12) is convex and the interior of the feasible set is non-empty, the

KKT conditions (14) are necessary and sufficient conditions for global optimality [49, Chap. 5]. Note that there is a unique primal optimal solution for (12), and so all KKT solutions of (14) share the same primal optimal solution.

B. Smoothness Properties of KKT Solutions

In order to determine the uniqueness, continuity, and differentiability properties of the KKT solution, we use the strong second-order sufficiency conditions (SSOSC) for optimality for the constrained optimization problem (12). Similar to unconstrained optimization problems, the SSOSC for constrained optimization requires strong convexity of the cost function at the primal optimal solution z^* along feasible directions. The SSOSC for (12) [50, Eq. (9)] states that for a KKT solution $(z^*(x), \lambda^*)$, $\exists a > 0$ such that

$$z^T \nabla_z^2 L(x, z^*(x), \lambda^*) z \geq a\|z\|_2^2, \tag{15}$$

$$\forall z \in \{z : z^{iT} \nabla_z A_k^i(x^i, z^{*i}(x)) = 0, \forall k \in \mathcal{J}^i(x^i, z^{*i}(x)),$$
$$z^{jT} \nabla_z A_k^j(x^j, z^{*j}(x)) = 0, \forall k \in \mathcal{J}^j(x^j, z^{*j}(x))\},$$

where $\mathcal{J}^i$ is defined in (11). The SSOSCs guarantee that the optimal primal solution $z^*(x)$ is a minimizer of the primal optimization problem (12), but can also be used to show smoothness properties. First, we show that for the minimum distance optimization problem (12), SSOSC holds under some conditions.

Lemma 5. *If $\mathcal{C}^i(x^i) \cap \mathcal{C}^j(x^j) = \emptyset$, then any KKT solution at x satisfies the strong second order sufficient conditions* (15).

Proof. The proof is provided in Appendix D. □

Now using the LICQ Assumption 3.c and the strong second-order sufficiency condition, we can show that there is a unique KKT solution for (14). Moreover, this KKT solution is also continuous, as shown next.

Lemma 6. *If Assumption 3 holds and $\mathcal{C}^i(x^i) \cap \mathcal{C}^j(x^j) = \emptyset$, then there is a unique KKT solution, written as $(z^*(x), \lambda^*(x))$, for* (14). *Moreover, the unique dual optimal solution $\lambda^*(x)$ is a continuous function of $x \in \mathcal{X}$.*

Proof. Linear independence condition holds at the primal optimal solution $z^*(x)$ by Assumption 3.c. If $\mathcal{C}^i(x^i) \cap \mathcal{C}^j(x^j) = \emptyset$, Lem. 5 shows that SSOSC holds at x. Finally, Assumption 3.a says that $A(x, z)$ is twice continuously differentiable. Then, [50, Thm. 2] shows that there is a unique dual optimal solution $\lambda^*(x)$ and that the unique KKT solution is continuous. □

Observe that the KKT conditions (14a) and (14b) are $2l + r^i + r^j$ equality constraints, in terms of the parameter x, for the KKT solution $(z^*(x), \lambda^*(x)) \in \mathbb{R}^{2l+r^i+r^j}$. If the Jacobian of these equality constraints with respect to the KKT solution is invertible, we can use the implicit function theorem to guarantee continuous differentiability of the KKT solution $(z^*(x), \lambda^*(x))$ in terms of x. However, we also need to make sure that the inequalities (14c) and (14d) are satisfied in the vicinity of x. We now state the result, which guarantees the continuous differentiability of the KKT solution at x.

Lemma 7. *If $\mathcal{C}^i(x^i) \cap \mathcal{C}^j(x^j) = \emptyset$ and the KKT solution $(z^*(x), \lambda^*(x))$ satisfies the strict complementary slackness condition, then for x' in a neighbourhood of x, the unique KKT solution $(z^*(x'), \lambda^*(x'))$ is continuously differentiable.*

Proof. The KKT solution is unique for x by Lem. 6. In order to show the continuous differentiability property for the KKT solution, we make use of [50]. The assumptions in [50, Thm. 1] are satisfied by Assumption 3.c, Lem. 5, and Assumption 3.a respectively. Note that second-order sufficient condition (SOSC) is weaker than SSOSC. Thus, using result (b) from [50, Thm. 1], for all x' in a neighbourhood of x the KKT solution $(z^*(x'), \lambda^*(x'))$ is continuously differentiable. □

The derivatives of the KKT solution (z^*, λ^*) with respect to x' in a vicinity of x can be computed by differentiating the KKT conditions (14) as done in [50] to obtain,

$$Q(x') \frac{\partial(z^*, \lambda^*)}{\partial x}(x') = V(x'), \tag{16}$$

where

$$Q(x') = \begin{bmatrix} \nabla_z^2 L & \nabla_z A^T \\ \text{diag}(\lambda^*) \nabla_z A & \text{diag}(A) \end{bmatrix} (x', z^*(x'), \lambda^*(x')), \tag{17}$$

is invertible when strict complementary slackness holds, and

$$V(x') = \begin{bmatrix} -\nabla_x \nabla_z L \\ -\text{diag}(\lambda^*) \nabla_x A \end{bmatrix} (x', z^*(x'), \lambda^*(x')). \tag{18}$$

Thus, whenever strict complementary slackness condition holds, (16) shows that $Dz^*(x) = [I_{2n}\ 0_{2n \times 2n}] Q(x)^{-1} V(x)$. This allows us to explicitly calculate the derivative of $h(x) = \|z^{*i}(x) - z^{*j}(x)\|_2^2$, as,

$$\frac{\partial h}{\partial x}(x) = 2(z^{*i}(x) - z^{*j}(x))^T \left(\frac{\partial z^{*i}}{\partial x}(x) - \frac{\partial z^{*j}}{\partial x}(x) \right). \tag{19}$$

However, when strict complementary slackness does not hold, $Q(x)$ is not invertible. For these border cases, we can still obtain the directional derivative of h as the solution of a system of inequalities. For the optimization problem (12), we first define the *active set* of constraints $\mathcal{J}_0^i$ and the *strictly active set* of constraints $\mathcal{J}_1^i$ for Robot i (and similarly for j) at a KKT solution $(z^*(x), \lambda^*)$ as

$$\mathcal{J}_0^i(x) = \{k \in [r^i] : A_k^i(x^i, z^{*i}(x)) = 0\}, \tag{20a}$$
$$\mathcal{J}_1^i(x) = \{k \in [r^i] : \lambda_k^{*i} > 0\}, \tag{20b}$$
$$\mathcal{J}_2^i(x) = \mathcal{J}_0^i(x) \setminus \mathcal{J}_1^i(x). \tag{20c}$$

We also adopt the following notation: If A^i has r^i rows and A^j has r^j rows, then the index for $A = [A^{iT}, A^{jT}]^T$ is obtained from the set $[r] := [r^i] \sqcup [r^j]$, where $\sqcup$ denotes the disjoint union, and for $(i, k) \in [r]$, $A_{(i,k)} := A_k^i$.

By the complementary slackness condition (14b), $\mathcal{J}_1^i(x) \subseteq \mathcal{J}_0^i(x)$, with the equality holding only when strict complementary slackness condition holds. When strict complementary slackness does not necessarily hold, we can use the following result to compute the derivative of h.

Theorem 1. *(Directional derivative of h) [50, Thm. 4] Let $\mathring{x} \in \mathbb{R}^{2n}$ be a direction of perturbation from $x \in \mathcal{X}$. Consider the following set of equations and inequalities for $(\mathring{z}, \mathring{\lambda})$,*

$$\nabla_z^2 L \mathring{z} + \mathring{\lambda} \nabla_z A = -\nabla_x \nabla_z L \mathring{x}, \tag{21a}$$
$$\nabla_z A_k \mathring{z} = -\nabla_x A_k \mathring{x}, \quad k \in \mathcal{J}_1(x), \tag{21b}$$
$$\nabla_z A_k \mathring{z} \leq -\nabla_x A_k \mathring{x}, \quad k \in \mathcal{J}_2(x),$$
$$\mathring{\lambda}_k = 0,\ k \in \mathcal{J}_0(x)^c, \quad \mathring{\lambda}_k \geq 0,\ k \in \mathcal{J}_2(x), \tag{21c}$$
$$\mathring{\lambda}_k (\nabla_z A_k \mathring{z} + \nabla_x A_k \mathring{x}) = 0, \quad k \in \mathcal{J}_2(x), \tag{21d}$$

where $(\cdot)^c$ represents the complement of a set. The set of equations and inequalities (21) *are evaluated at* $(x, z^*(x), \lambda^*)$ *and the index set* $\mathcal{J}_0(x) = \mathcal{J}_0^i(x) \sqcup \mathcal{J}_0^j(x)$ *(and similarly for* $\mathcal{J}_1(x)$ *and* $\mathcal{J}_2(x)$*).*

Then, (21) *has a unique solution,* $(\mathring{z}^*, \mathring{\lambda}^*)$*, which is the directional derivative of* (z^*, λ^*) *at* x *along* $\mathring{x}$*.*

Thm. 1 provides a method to calculate the directional derivative of z^* along a direction $\mathring{x}$ by solving for the feasible solution $\mathring{z}$ of (21). Moreover, when strict complementary slackness holds, (21) reduces to (16).

To summarize, Thm. 1 provides a set of constraints whose feasible solution for a given direction $\mathring{x}$ provides the directional derivative $(\mathring{z}, \mathring{\lambda})$ of the KKT solution (z^*, λ^*). In the next section, we use the directional derivative $\mathring{z}$ along a direction $\mathring{x}$, determined by the system inputs, to calculate $\dot{h}$ using (19) and enforce the NCBF constraint (7) for strictly convex sets.

V. Obstacle Avoidance for Strictly Convex Sets

The results in Lem. 7 and Thm. 1 allow us to find the directional derivatives of the optimal solution $z^*(x)$ in terms of $\dot{x}$, which is determined by the system dynamics and the input. Since $h(x) = \|z^*(x)\|_2^2$, we can find the directional derivatives of h, and thus enforce the NCBF constraint (7). Thus, using (19) and the set of constraints (21), we can enforce the NCBF constraint (7) in terms of the control input u, i.e., we can design a control law $u(x)$ which guarantees the NCBF constraint pointwise.

However, the constraints (21) depend on the index sets $\mathcal{J}_0, \mathcal{J}_1$, and $\mathcal{J}_2$, which can change with x, meaning the $(\mathring{z}, \mathring{\lambda})$ obtained from (21) may not be continuous. The resulting control law can be discontinuous, and the closed-loop trajectory would then be the Filippov solution corresponding to the closed-loop system. So we must ensure that the Filippov solution satisfies the NCBF constraint.

To do this, we use *almost active set* of constraints similar to [51, Def. 7]. We define the almost active set of constraints, for any $\epsilon > 0$, as

$$\mathcal{J}_{2,\epsilon}(x) = \{k \in [r] : \lambda_k^*(x) < \epsilon, A_k(x, z^*(x)) > -\epsilon\}. \quad (22)$$

Note that $\mathcal{J}_{2,\epsilon}^i(x) \supseteq \mathcal{J}_2^i(x)$, and $\mathcal{J}_0^c(x) \cap \mathcal{J}_{2,\epsilon}$ and $\mathcal{J}_1(x) \cap \mathcal{J}_{2,\epsilon}$ may be non-empty.

Using the expression (19) as the basis, we propose that the following optimization problem can be used to enforce the NCBF constraint between Robots i and j:

$$\begin{aligned}
u^*(x) = &\operatorname*{argmin}_{\{u, \mathring{z}, \mathring{\lambda}\}} \|u - u^{nom}(x)\|^2 && \text{(23a)}\\
\text{s.t.}\quad & 2(z^{*i}(x) - z^{*j}(x))(\mathring{z}^i - \mathring{z}^j) \geq -\alpha \cdot h(x), && \text{(23b)}\\
& \mathring{x} = f(x) + g(x)u, && \text{(23c)}\\
& \nabla_z^2 L \mathring{z} + \mathring{\lambda} \nabla_z A + \nabla_x \nabla_z L \mathring{x} = 0, && \text{(23d)}\\
& \nabla_z A_k \mathring{z} = -\nabla_x A_k \mathring{x}, \quad k \in \mathcal{J}_1(x), && \text{(23e)}\\
& \nabla_z A_k \mathring{z} \leq -\nabla_x A_k \mathring{x}, \quad k \in \mathcal{J}_{2,\epsilon}(x), && \\
& \mathring{\lambda}_k = 0,\ k \in \mathcal{J}_0(x)^c, \quad \mathring{\lambda}_k \geq 0,\ k \in \mathcal{J}_{2,\epsilon}(x), && \text{(23f)}\\
& \mathring{\lambda}_k(\nabla_z A_k \mathring{z} + \nabla_x A_k \mathring{x}) = 0, k \in \mathcal{J}_{2,\epsilon}(x), && \text{(23g)}\\
& \|\mathring{z}\| \leq M, \quad \|\mathring{\lambda}\| \leq M, && \text{(23h)}\\
& u \in \mathcal{U}. && \text{(23i)}
\end{aligned}$$

where $u^{\text{nom}}(x)$ is a nominal feedback controller not designed with safety in consideration and can be obtained using a

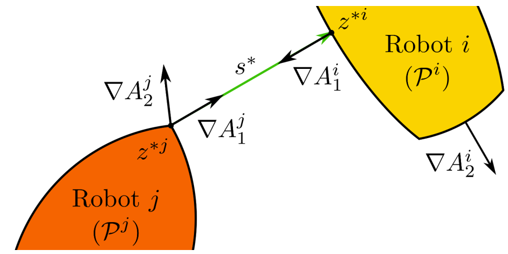


Fig. 3: For a given state x, the figure illustrates the separating vector $s^* = z^{*i} - z^{*j}$ in green. For strictly convex sets there is a unique optimal solution $z^* = (z^{*i}, z^{*j})$, and the gradients of the constraints, ∇A_1^i at z^{*i} and ∇A_1^j and ∇A_2^j at z^{*j} are shown. The KKT condition (14a) indicates that a conic combination of ∇A_k^j must be equal to s^* (the dual variables are the coefficients). From the figure, we can see that s^* lies in the cone generated by ∇A_1^j and ∇A_2^j (and similarly for i), and that $\lambda_2^{*j} = \lambda_2^{*i} = 0$. Thus the index sets (set $\mathcal{J}_0$ of primal active and $\mathcal{J}_1$ of dual inactive constraints) at the state x are $\mathcal{J}_0^i = \mathcal{J}_1^i = \{1\}, \mathcal{J}_0^j = \{1, 2\}$, and $\mathcal{J}_1^j = \{1\}$. The combined index sets can be written as: $\mathcal{J}_0 = \{(i, 1), (j, 1), (j, 2)\}$, and $\mathcal{J}_1 = \{(i, 1), (j, 1)\}$.

Lyapunov function or a tracking control law. Note that $\mathring{x}$ represents the derivative of x when an input u is applied, and $\mathring{z}$, when feasible for (23d)-(23h), is the derivative of z^* at x along $\mathring{x}$ The constraint (23b) is the NCBF constraint $\dot{h}(x) \geq -\alpha \cdot h(x)$, (23c) is the dynamics constraint for the system. The constraints (23d)-(23g) are the same as (21), except the active constraint set $\mathcal{J}_2(x)$ is replaced by the almost active constraint set $\mathcal{J}_{2,\epsilon}(x)$. The constraint (23h) restricts the norm of the derivatives with M being a large number.

Remark 2. *Whenever strict complementarity holds, we can directly compute the gradient of* z^* *using* (16) *and compute* $\frac{\partial h}{\partial x}$ *as a function of* x *and* u*, without having to add the additional constraints* (23d)-(23h)*. This will result in a significantly smaller quadratic program, leading to faster computational times. When the convex set is defined by a single constraint (such as for ellipsoids [40]), strict complementarity always holds, and* (16) *can be used directly. When strict complementary slackness does not hold, we must use* (23)*. In practice, strict complementary slackness almost always holds, and* (16) *can be used directly.*

We can now prove that the control law obtained from (23) guarantees safety for the closed-loop system.

Theorem 2. *(Safety for strictly-convex sets) Let* $x(0) \in \mathcal{S}$ *and* (23) *be feasible* $\forall\ x \in \mathcal{S}$*, with some* $\alpha > 0$*. Then, any measurable feedback control law obtained as a solution of* (23) *makes the closed-loop system safe. The system's safety is independent of the cost function used in* (23)*.*

Proof. The proof is provided in Appendix E. □

VI. Smoothness Properties for Polytopes with Dual Analysis

In the previous two sections, we determined the smoothness properties of the KKT solutions for strictly convex sets and used them to enforce the NCBF constraint. We repeat this procedure for polytopes in the next two sections. We follow the outline of our previous paper [1]. In this section, we detail the continuity and differentiability properties for polytopes, which will then be used to derive the NCBF formulation for polytopes in the next section. This section aims to determine a set of smooth, explicit constraints similar to (21), which will allow us to enforce the NCBF constraints for polytopes. The discussion in this section deviates from Sec. IV since the optimal solution z^* might be discontinuous for polytopes, and thus polytopic geometries are not amenable to the same techniques used for strictly convex sets. Instead, we will use the dual formulation.

Consider a class of robots with polytopic geometries. For Robot i at the state $x^i \in \mathcal{X}^i$, let $\mathcal{C}^i(x^i)$ be the l-dimensional region associated to the robot, defined by

$$\mathcal{C}^i(x^i) := \{z \in \mathbb{R}^l : A^i(x^i)z - b^i(x^i) \leq 0\}, \tag{24}$$

where $A^i : \mathcal{X}^i \to \mathbb{R}^{r^i \times l}$ and $b^i : \mathcal{X}^i \to \mathbb{R}^{r^i \times 1}$ represent the half spaces that specify the domain boundary. Polytopes are convex and closed by definition, and the set of polytopic sets is disjoint from the set of strictly convex sets.

Assumption 4. *We assume the following for all i:*

(a) A^i, b^i are continuously differentiable on $\mathcal{X}^i$.

(b) The set of active constraints at any point of $\mathcal{C}^i(x^i)$ are linearly independent for all $x^i \in \mathcal{X}^i$, i.e. the set of normal vectors $\{A^i_k(x^i) : k \in \mathcal{J}(x^i, z^i)\}$ is linearly independent $\forall z^i \in \mathcal{C}^i(x^i)$ and $x^i \in \mathcal{X}^i$. In practice, we only need to check this condition at the vertices of $\mathcal{C}^i(x^i)$.

(c) $\mathcal{C}^i(x^i)$ is bounded, and thus compact, and has a non-empty interior for all $x^i \in \mathcal{X}^i$.

Consider the minimum distance optimization problem (8) for polytopes as,

$$\begin{aligned} h(x) := \min_{z} \quad & \|z^i - z^j\|_2^2 \\ \text{s.t.} \quad & A^i(x^i)z^i \leq b^i(x^i),\ A^j(x^j)z^j \leq b^j(x^j), \end{aligned} \tag{25}$$

Assumption 4.b requires that at most l planes can define any vertex of the polytope and is equivalent to the LICQ Assumption 3.c for strictly convex sets. Any polytope that does not satisfy Assumption 4.b can be tessellated into smaller polytopes, such as tetrahedra, which do satisfy the assumption. Once tessellated, the method described in this section can be applied to each of these polytopes.

Assumption 4 and the results from Sec. III can be used to determine the smoothness properties of the minimum distance between polytopic sets. Similar to strictly convex sets, Assumption 4 implies that Assumption 2 holds. Thus, the minimum distance between polytopic sets is locally Lipschitz continuous and a candidate NCBF. In particular, Assumption 4.c guarantees regularity conditions are satisfied for (25), which are necessary and sufficient conditions for the existence of an optimal solution [52, Thm. 2.1]. Moreover, regularity is sufficient to establish the continuity [52, Thm. 2.3] and directional differentiability [52, Thm. 2.4] of the minimum distance h between $\mathcal{C}^i(x^i)$ and $\mathcal{C}^j(x^j)$.

Polytopes are convex sets, and so by Lem. 2, there is a unique separating vector $s^*(x)$ for (25). However, multiple optimal primal solutions can exist since polytopes are not strictly convex. Let $\mathcal{Z}^*(x)$ be the set of all the pairs of optimal solutions, (z^{*i}, z^{*j}), at x. Note that $\mathcal{Z}^*(x)$ is a convex, compact, and non-empty set.

Although $h(x)$ is locally Lipschitz continuous, the primal optimal solutions $z^{*i}(x)$, $z^{*j}(x)$ could be discontinuous or even non-unique. So, we cannot represent $\dot{h}(t)$ using the derivatives of the primal and dual optimal solutions, as we did in Sec. IV-B. However, we can obtain a lower bound on $\dot{h}(t)$ using the dual program.

A. Dual Program and KKT Conditions

The dual program of a minimization problem is a maximization problem in terms of the corresponding dual variables. For the minimum distance optimization problem between polytopes, we can obtain the dual program in terms of only the dual variables λ, thus bypassing the need for primal variables. Moreover, for quadratic programs, such as (25), the dual program has the same optimal solution as that of (25).

Lemma 8. *[1, Lem. 4] The dual program corresponding to* (25) *is:*

$$\begin{aligned} h(x) = \max_{\{\lambda^i, \lambda^j\}} \quad & \Lambda(x, \lambda) \\ \textit{s.t.} \quad & \lambda^i A^i(x^i) + \lambda^j A^j(x^j) = 0, \\ & \lambda^i, \lambda^j \geq 0, \end{aligned} \tag{26}$$

where Λ is the Lagrangian dual function, given by

$$\Lambda(x, \lambda) = -\frac{1}{4}\|\lambda^i A^i(x^i)\|_2^2 - \lambda^i b^i(x^i) - \lambda^j b^j(x^j). \tag{27}$$

Similar to the case of strictly convex sets, the KKT conditions for polytopes state that for z^* to be optimal for (25), there must exist a pair (z^*, λ^*) that satisfies,

$$\begin{bmatrix} 2(z^{*i} - z^{*j}) + A^i(x^i)^T \lambda^{*iT} \\ 2(z^{*j} - z^{*i}) + A^j(x^j)^T \lambda^{*jT} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{28a}$$

$$\lambda^{*i}(A^i(x^i)z^{*i} - b^i(x^i)) = 0, \tag{28b}$$

$$\lambda^{*j}(A^j(x^j)z^{*j} - b^j(x^j)) = 0,$$

$$\lambda^{*i} \geq 0,\ \lambda^{*j} \geq 0, \tag{28c}$$

$$A^i(x^i)z^{*i} \leq b^i(x^i),\ A^j(x^j)z^{*j} \leq b^j(x^j), \tag{28d}$$

where (28a) is the stationarity condition along the set feasible directions, (28c) are the non-negativity conditions for the dual variables, and (28d) are the primal feasibility conditions for z^*. The conditions (28b) are called complementary slackness conditions. The dual program (26) and the KKT conditions (28) can be used to determine the smoothness properties of the dual solution.

B. Smoothness properties of Dual Solutions

Since the minimum distance problem for polytopes can have multiple optimal solutions, we first redefine the index sets for polytopes. For all $(z^{*i}, z^{*j}) \in \mathcal{Z}^*(x)$, define $\mathcal{J}_0^i(x) \subset [r^i]$ ($\mathcal{J}_0^j(x) \subset [r^j]$) as the set of indices of constraints that are active for all optimal solutions z^{*i} ($z^{*j}(x)$) for $\mathcal{C}^i(x)$ ($\mathcal{C}^j(x)$), i.e.,

$$\mathcal{J}_0^i(x) := \{k \subset [r^i] : k \in \mathcal{J}^i(x^i, z^{*i})\ \forall z^* \in \mathcal{Z}^*(x)\}, \tag{29}$$

$$\mathcal{J}_0^j(x) := \{k \subset [r^j] : k \in \mathcal{J}^j(x^j, z^{*j})\ \forall z^* \in \mathcal{Z}^*(x)\},$$

where $\mathcal{J}^i$ is defined in (11). By convexity of $\mathcal{Z}^*(x)$, there exists some optimal solution, represented by $\bar{z}^* \in \mathcal{Z}^*(x)$, such that $\mathcal{J}^i(x^i, \bar{z}^{*i}) = \mathcal{J}_0^i(x)$ and $\mathcal{J}^j(x^j, \bar{z}^{*j}) = \mathcal{J}_0^j(x)$ (see [53, Rem. 3.16]). Then,

$$\text{Aff}^i(x) := \{z^i : A^i_{\mathcal{J}_0^i(x)}(x)z^i = b^i_{\mathcal{J}_0^i(x)}(x)\}, \tag{30}$$

$$\text{Aff}^j(x) := \{z^j : A^j_{\mathcal{J}_0^j(x)}(x)z^j = b^j_{\mathcal{J}_0^j(t)}(x)\},$$

represent two parallel affine spaces such that the minimum distance between $\mathcal{C}^i(x)$ and $\mathcal{C}^j(x)$ is the distance between the affine spaces $\text{Aff}^i(x)$ and $\text{Aff}^j(x)$. This is because a KKT solution $(\bar{z}^*, \lambda^*)$ of (25) is also a KKT solution for the minimum distance optimization problem between $\text{Aff}^i(x)$ and $\text{Aff}^j(x)$. These affine spaces can be points, hyperplanes, or even the entire space if the two polytopes intersect. An example is depicted in Fig. 4.

For convenience, we choose to work in the time domain for the rest of the section. Let $x(t), t \in [0, T]$ be the Filippov solution corresponding to some measurable feedback control law $u : \mathcal{X} \to \mathcal{U}$. We can now write all functions of x as functions of time explicitly, such as, $h(t) := h(x(t))$, $\mathcal{Z}^*(t) := \mathcal{Z}^*(x(t))$, $s^*(t) = s^*(x(t))$ etc. We also make the following assumption about the geometry of the affine spaces $\text{Aff}^i(t)$ and $\text{Aff}^j(t)$.

Assumption 5. *For almost all $t \in [0, T]$, $\exists\ \epsilon > 0$ such that dim($\text{Aff}^i(\tau)$), dim($\text{Aff}^j(\tau)$), and the dimension of the othogonal subspace common between $\text{Aff}^i(\tau)$ and $\text{Aff}^j(\tau)$ are constant for $\tau \in [t, t + \epsilon)$.*

Remark 3. *Assumption 5 is true when the set of times when states of the system oscillate infinitely fast has zero-measure. In practice, the states of the system do not oscillate infinitely fast due to limited control frequency (due to sample-and-hold control implementation [45, Eq. (36)]) and the inertia of the system.*

Further, we can use the KKT conditions (28) and the dual program (26) to determine the properties of the dual optimal solution.

Lemma 9. *Under Assumption 4, a unique dual optimal solution $\lambda^*(t)$ of (26) exists, and $\lambda^*(t)$ is continuous for all $t \in [0, T]$. Further, under Assumption 5, the dual optimal solution $\lambda^*(t)$ is right-differentiable for almost all $t \in [0, T]$.*

Proof. The proof is provided in Appendix F. □

Finally, similar to Thm. 1, we can explicitly write a linear program for computing a lower bound of $\dot{h}(t)$ by differentiating the cost and constraints of (26) as,

Lemma 10. *(Lower Bound for Derivative of h as an LP) Let,*

$$g(t) = \max_{\{\dot{\lambda}\}} \quad \dot{\Lambda}(t, \lambda^*(t), \dot{\lambda}) \tag{31a}$$

$$\text{s.t.} \quad \dot{\lambda}^i A^i(t) + \lambda^{*i}(t)\dot{A}^i(t) + \dot{\lambda}^j A^j(t) + \lambda^{*j}(t)\dot{A}^j(t) = 0, \tag{31b}$$

$$\dot{\lambda}^i_k \geq 0 \quad \text{if } \lambda^{*i}_k(t) = 0, \qquad \dot{\lambda}^j_k \geq 0 \quad \text{if } \lambda^{*j}_k(t) = 0. \tag{31c}$$

where $\dot{\Lambda}(t, \lambda, \dot{\lambda})$ represents the time-derivative of Lagrangian dual function $\Lambda(x(t), \lambda)$ and is defined as follows,

$$\dot{\Lambda}(t, \lambda, \dot{\lambda}) = -\frac{1}{2}\dot{\lambda}^i A^i(t) A^i(t)^T \lambda^{i\,T} - \frac{1}{2}\lambda^i A^i(t)\dot{A}^i(t)^T \lambda^{i\,T} - \dot{\lambda}^i b^i(t) - \lambda^i \dot{b}^i(t) - \dot{\lambda}^j b^j(t) - \lambda^j \dot{b}^j(t). \tag{32}$$

Then, for almost all $t \in [0, T]$, $\dot{h}(t) \geq g(t)$.

Proof. The proof is provided in Appendix G. □

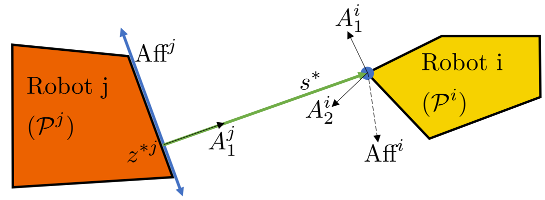


Fig. 4: For a given state x, the figure illustrates the separating vector s^* in green. The vectors A^i_k represent the normal vectors of the hyperplanes defining the $\mathcal{C}^i(x^i)$. The set of primal optimal solutions is a singleton set given by $\mathcal{Z}^* = \{(z^{*i}, z^{*j})\}$. Although z^* is unique for the configuration depicted, this is not necessarily the case for all states, such as when the planes defined by A^j_1 and A^i_2 are parallel. Similar to the strictly convex set case, s^* must lie in the conic set generated by A^i_1 and A^i_2 (and similarly for j). The coefficients of this conic combination are the dual variables λ^*. The index sets for the state x can be written as: $\mathcal{J}^i_0 = \{1, 2\}$, and $\mathcal{J}^j_0 = \{1\}$. The minimum distance between any two polytopes is the same as the minimum distance between the affine sets Aff^i and Aff^j, represented in blue.

VII. Obstacle Avoidance for Polytopes

The result from Lem. 10 allows us to write $\dot{h}(t)$ in terms of the time derivatives of A^i and A^j, which in turn depend on the inputs.

Based on the LP (31), we can implement the NCBF constraint by enforcing, for some $(\dot{\lambda}^i, \dot{\lambda}^j)$ feasible for (31),

$$\dot{\Lambda}(t, \lambda^*(t), \dot{\lambda}) \geq -\alpha \cdot h(t). \tag{33}$$

Lem. 10 then guarantees that

$$\dot{h}(t) \geq \dot{\Lambda}(t, \lambda^*(t), \dot{\lambda}) \geq -\alpha \cdot h(t), \tag{34}$$

holds for almost all $t \in [0, T]$, which is the required NCBF constraint.

Remark 4. *In order to express the NCBF constraint as in (34), $\dot{h}(t)$ needs to be expressed as a maximization problem. This is the primary motivation for considering the dual problem, since writing $\dot{h}(t)$ using the primal problem, similar to (31), would result in a minimization problem.*

We can use (33) to motivate a feedback control law to guarantee the safety of the system. The input u implicitly affects $\dot{\Lambda}$ via the derivatives of the matrices A^i, A^j, b^i, b^j. Note that by (32), $\dot{\Lambda}$ is affine in $\dot{\lambda}$, and u. To implement the safety-critical control law, $\forall x \in \mathcal{S}$, (26) is first used to compute $h(x)$ and $\lambda^*(x)$, and then the optimal solution of following quadratic program is used as the feedback control.

$$u^*(x) = \underset{\{u, \dot{\lambda}\}}{\text{argmin}} \; \|u - u^{nom}(x)\|^2 \tag{35a}$$

$$\text{s.t.} \quad \dot{\Lambda}(t, \lambda^*(x), \dot{\lambda}, u) \geq -\alpha \cdot h(x) \tag{35b}$$

$$\dot{\lambda}^i A^i(x) + \lambda^{*i}(x)(\mathcal{L}_{f^i} A^i(x) + \mathcal{L}_{g^i} A^i(x) u) + \lambda^{*j}(x)(\mathcal{L}_{f^j} A^j(x) + \mathcal{L}_{g^j} A^j(x) u) = -\dot{\lambda}^j A^j(x) \tag{35c}$$

$$\dot{\lambda}^i_k \geq 0 \;\text{ if }\; \lambda^{*i}_k(x) < \epsilon, \qquad \dot{\lambda}^j_k \geq 0 \;\text{ if }\; \lambda^{*j}_k(x) < \epsilon, \tag{35d}$$

$$|\dot{\lambda}^i| \leq M, |\dot{\lambda}^j| \leq M, \tag{35e}$$

$$u \in \mathcal{U}, \tag{35f}$$

where $\mathcal{L}_{(\star)}(\cdot)$ represents the Lie derivative of $(\cdot)$ along $(\star)$, $\epsilon > 0$ is a small constant, and M is a large number.

Theorem 3. *(Safety for polytopic sets) Let $x(0) \in \mathcal{S}$ and (35) be feasible $\forall\ x \in \mathcal{S}$. Then, the feedback control law (35) makes the closed-loop system safe, irrespective of the cost function.*

Proof. The proof is provided in Appendix H. □

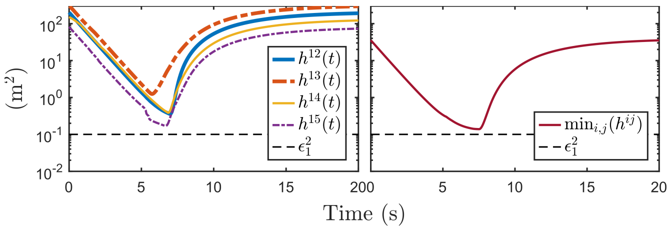


Fig. 5: Square of the minimum distance (NCBF) between different pairs of strictly convex-shaped robots, corresponding to Fig. 1b. The left figure shows the NCBF between robot 1 (in blue) and all other robots, while the right figure plots the minimum of all 20 pairwise NCBFs as a function of time. Since the NCBFs are greater than the safety margin ϵ_1^2, the robots don't collide with each other, and safety is maintained. Both plots have log scales on the y-axis.

Remark 5. *In practice, $h(x)$ is redefined as $(h(x)-\epsilon_1^2)$, where $\epsilon_1 > 0$ is a small number. This prevents the gradient of $h(x)$ from becoming zero at the boundary of the safe set [1, Rem. 3]. Moreover, if $h(x)$ is redefined as $(h(x)-\epsilon_1^2)$ then the term $\alpha \cdot h(t)$ in the NCBF constraint* (7) *can be replaced by $\alpha(h(t)-\epsilon_1^2)$, where α is any extended class-K function [4].*

Remark 6. *Both the formulations* (35) *and* (23) *can be extended to multiple robots by introducing the corresponding optimization variables for each pair of robots, which will also be validated in the Sec. VIII. For the case of polytopes, each pair of robots would have separate dual variables $\mathring{\lambda}$, which will be used to enforce the NCBF constraint between that pair of robots. Since safety is maintained pairwise, it is also maintained for the whole system.*

VIII. Results

In this section, we provide simulation results to demonstrate obstacle avoidance between strict convex-shaped robots in real-time, using the formulation derived in Sec. IV. Additional simulation results for the polytope-shaped robots, using the formulation (35), can be found in [1].

A. Simulation Setup

The simulation environment consists of 5 strict convex-shaped robots, as shown in Fig. 1b. The robots in the figure are labeled 1 to 5 going counter-clockwise, starting with the blue robot. Robot 1 is described by the shape $\{z : (z_1/1.5)^4 + z_2^4 \leq 1\}$, Robot 5 by $\{z : (z_1/2)^6 + z_2^6 \leq 1\}$, and Robot 4 is an ellipse. Robots 2 and 3 are formed by the intersection of three and two circles. All these shapes satisfy the assumptions for strictly convex sets. The states of all five robots are $x = (x_1, x_2, x_3) \in \mathbb{R}^2 \times \mathcal{S}^1$, where (x_1, x_2) is the geometric center of the robot and x_3 is the angle of rotation. Robots 1, 2, and 3 are assumed to have three inputs, $u = (v_1, v_2, \omega)$ and integrator dynamics as

$$\dot{x}_1 = v_1, \quad \dot{x}_2 = v_2, \quad \dot{x}_3 = \omega, \tag{36}$$

where (v_1, v_2) is the velocity and ω is the angular velocity. The inputs to Robots 4 and 5 are $u = (v, \omega)$ corresponding to unicycle dynamics,

$$\dot{x}_1 = v\cos(x_3), \quad \dot{x}_2 = v\sin(x_3), \quad \dot{x}_3 = \omega, \tag{37}$$

where v is the velocity along x_3 and ω is the angular velocity. The initial positions for the robots are chosen along the boundary of an ellipse with a semi-major axis length of $15\,\mathrm{m}$ and a semi-minor axis length of $7.5\,\mathrm{m}$, and the final positions are diametrically opposite to the initial positions. The nominal controller for the integrator is obtained using a Proportional control law, while that of the unicycle is derived from a CLF [54].

The NCBF is chosen as the square of the minimum distance as described in (12). Note that, for any safe configuration of the robots, by choosing $u = 0$, $\mathring{z} = 0$, and $\mathring{\lambda} = 0$, the optimization problem (23) can be made feasible, thus satisfying the assumptions of Thm. 2. The value of the margin ϵ_1, as defined in Rem. 5 is chosen as $\epsilon_1^2 = 0.1\,\mathrm{m}^2$.

B. Simulation Results

The simulations are performed on a laptop with an 8-core $2.20\,\mathrm{GHz}$ Intel Core i7 processor using MATLAB. For strictly convex-shaped robots, the nonlinear distance optimization (12) is solved using *fmincon* with warm start, and the control optimization (23) is solved using *quadprog*. For polytope-shaped robots, both the distance (25) and control (35) optimizations are solved using *quadprog*. The trajectory of the five strictly convex-shaped robots is illustrated in Fig. 1b using snapshots. Similarly, snapshots of five polytope-shaped robots are shown in Fig. 1a. The NCBF h^{ij} is greater than ϵ_1^2 for all 20 pairs of robots as depicted in Fig. 5, and thus the safety of the system is maintained.

TABLE II: Statistics of the computation time (ms) per iteration for strictly convex sets

Timing (ms)	*mean* $\pm$ *std*	*p50*	*p99*	*max*
Distance OPTs (12)	4.80 $\pm$ 1.50	4.60	8.50	44.0
NCBF OPT (23)	1.00 $\pm$ 2.60	0.70	4.00	51.9
Total (20+1 = 21 OPTs)	88.5 $\pm$ 14.9	85.3	136	234

The computation times for calculating the 20 pairwise minimum distances, the control inputs for each robot, and the total computation time are provided in Tab. II. *p50* and *p99* refer to the 50-th and 99-th percentile values. There are some large outliers in the computation times of the distance and control optimizations, but they infrequently occur, as indicated by the 99-th percentile values. The values in the table indicate that the controller can be run at around $10\,\mathrm{Hz}$ for five robots. Moreover, if the pairwise distances are computed parallelly, the controller can be run in real-time. Note that for N robots, we need to compute $N(N+1)/2$ pairwise distances, the computation time for which grows quadratically with N. Distributing the distance computation will result in linear growth of the computation time.

C. Improvements

The following techniques can be used to reduce the computation times of the NCBF formulation.

1) NCBF enforcement region: As seen in the previous subsection, pairwise distance computations increase quadratically with the number of robots. One way to address this issue is to enforce the NCBF constraint only between those pairs of robots that are close enough. This bounds the number of NCBF constraints at any given time and improves the computation times. With a large enough NCBF enforcement radius, system safety can be maintained.

2) Projection of constraints to input space: Note that when the index set $\mathcal{J}_2(x) = \emptyset$, the formulation (23) becomes a QP. In this case, we can project the linear constraints from the $(u, \mathring{z}, \mathring{\lambda})$-space to the u-space using Fourier-Motzkin Elimination (FME) [53, Sec. 3.1]. Thus instead of combining all the constraints from each pair of robots, as described in Rem. 6, we can compute the projected constraints and solve the resulting QP entirely in the input space. This results in a decrease in the number of variables in the safety optimization from $\mathcal{O}(N^2)$ to $\mathcal{O}(N)$.

IX. Conclusion

In this paper, we presented a general framework for non-conservative obstacle avoidance between strictly convex-shaped regions as well as between polytopic-shaped regions using the KKT solutions of their minimum distance optimization problems. We showed that our method's control input can be computed using a QP for systems with control affine dynamics, enabling real-time implementation. We validated our algorithm with real-time obstacle avoidance tasks for multi-robot systems with strictly convex-shaped or polytopic robots. We also established theoretical results on the smoothness of the minimum distance function and the KKT solutions of the minimum distance optimization problem, using which we proved system safety.

Appendix

Proofs for Strictly Convex Sets

A. Proof of Lem. 2

Since the Minkowski sum of convex, compact sets is also convex and compact, the optimization problem (8) is equivalent to (9). So (9) is a convex optimization problem with a strictly convex cost function and compact, non-empty feasible set. Thus there exists a unique minimizer s^* of (9), such that $h(x) = \|s^*\|_2^2 = \|z^{*i} - z^{*j}\|_2^2$, for all optimal solutions z^*.

Now let $\mathcal{C}^i(x^i)$ be strictly convex and (z^{*i1}, z^{*j1}) and (z^{*i2}, z^{*j2}) be two different optimal solutions. Since $\mathcal{C}^i(x^i) \cap \mathcal{C}^j(x^j) = \emptyset$, z^{*i1} and z^{*i2} lie on the boundary of $\mathcal{C}^i(x^i)$ (and similarly for j). By uniqueness of s^*, $z^{*i1} - z^{*j1} = z^{*i2} - z^{*j2} = s^*$. Since the set of optimal solutions for convex optimization problems is convex, $((z^{*i1} + z^{*i2})/2, (z^{*j1} + z^{*j2})/2)$ is also optimal, i.e. they correspond to the minimum distance between $\mathcal{C}^i(x^i)$ and $\mathcal{C}^j(x^j)$. However, this implies that $(z^{*i1} + z^{*i2})/2$ lies on the boundary of $\mathcal{C}^i(x^i)$, which is not possible since $\mathcal{C}^i(x^i)$ is strictly convex. Thus (8) has a unique optimal solution z^*.

B. Proof of Lem. 3

We first state the Maximum Theorem, rewritten in the context of the minimum distance between sets, which is then used to prove the continuity property.

Theorem 4. *(Maximum Theorem) [55, Pg. 306] Let $\rho^{ij} : \mathcal{Z} \times \mathcal{X} \to \mathbb{R}$ be a continuous function and $\mathcal{C}^{ij} : \mathcal{X} \to 2^{\mathcal{Z}}$ be a compact-valued map such that $\mathcal{C}^{ij}(x) \neq \emptyset\ \forall x \in \mathcal{X}$. Define the function $\rho^{*ij} : \mathcal{X} \to \mathbb{R}$ as,*

$$\rho^{*ij}(x) := \inf_{s \in \mathcal{Z}} \{\rho^{ij}(s, x) : s \in \mathcal{C}^{ij}(x)\}, \tag{38}$$

and the set of minimizers as,

$$\mathcal{C}^{*ij}(x) := \operatorname*{argmin}_{s \in \mathcal{Z}} \{\rho^{ij}(s, x) : s \in \mathcal{C}^{ij}(x)\}. \tag{39}$$

If $\mathcal{C}^{ij}$ is lower and upper semi-continuous, then ρ^{*ij} is continuous and $\mathcal{C}^{*ij}$ is upper semi-continuous, non-empty, and compact-valued.

Let $\mathcal{Z} = \mathbb{R}^l$, and for some $s \in \mathcal{Z}$ and $x \in \mathcal{X}$, let $\rho^{ij}(s, x) = \|s\|_2^2$, which is a continuous function. Define $\mathcal{C}^{ij}(x) = \mathcal{C}^i(x^i) - \mathcal{C}^j(x^j)$, which is compact, convex, non-empty, and upper and lower semi-continuous (since $\mathcal{C}^i$ and $\mathcal{C}^j$ are upper and lower semi-continuous). Then the optimization problem (38) is the same as (9), and $\rho^{*ij}(x) = h(x)$ and $\mathcal{C}^{*ij}(x) = \{s^*(x)\}$, which is a singleton set for all x by Lem. 2. All the conditions for the maximum theorem are met, and thus h is continuous for all $x \in \mathcal{X}$ and $\mathcal{C}^{*ij}$ is upper semi-continuous.

Note that upper semi-continuity for a singleton set-valued function mapping to $2^{\mathbb{R}^l}$ is equivalent to the continuity of the function mapping to $\mathbb{R}^l$ [45, Sidebar 7]. So, by the maximum theorem, s^* is continuous for all $x \in \mathcal{X}$.

If $\mathcal{C}^i(x^i)$ is strictly convex, we can perform a similar computation. Let $\mathcal{Z} = \mathbb{R}^l \times \mathbb{R}^l$, and for some $z \in \mathcal{Z}; x \in \mathcal{X}$, $\rho^{ij}(z, x) = \|z^i - z^j\|_2^2$, which is continuous. Define $\mathcal{C}^{ij}(x) = (\mathcal{C}^i(x^i), \mathcal{C}^j(x^j))$, which is lower and upper semi-continuous, non-empty, and compact-valued by assumption. Then (38) is equivalent to (8) and $\mathcal{C}^{*ij}(x) = \{(z^{*i}(x), z^{*j}(x))\}$, which is a singleton set in a neighbourhood of x by Lem. 2 and continuity of h. Since all conditions for the maximum theorem are satisfied, $\mathcal{C}^{*ij}$ is upper semi-continuous and thus (z^{*i}, z^{*j}) is a continuous function of $x \in \mathcal{S}$.

C. Proof of Lem. 4

Consider the minimum distance optimization problem (8) with the convex sets as defined by (10), and let $\mathcal{Z}^*(x)$ be the set of all primal optimal solutions of (8). Since $\mathcal{C}^i(x^i)$ and $\mathcal{C}^j(x^j)$ are bounded, $\mathcal{Z}^*(x)$ is also bounded and non-empty (by Lem. 3). Choose at any $x \in \mathcal{X}$, by Assumption 2.c, $\mathcal{W} \supseteq \mathcal{Z}^*(x)$ such that $\mathcal{W}$ is open, convex, and bounded. The cost function of the optimization problem (8) is $\|z^i - z^j\|_2^2$, which is quadratic and thus Lipschitz continuous on $\mathcal{X} \times \mathcal{W}$, since $\mathcal{W}$ is bounded. Finally, by Assumption 2.c, $A_k^i(\cdot, z^i)$ and $A_k^j(\cdot, z^j)$ are Lipschitz continuous around x for each $z \in \mathcal{W}$ and k, with a Lipschitz constant independent of $z \in \mathcal{W}$. By [47, Thm. 1], $h(x)$ is Lipschitz continuous around x.

D. Proof of Lem. 5

The KKT conditions are necessary and sufficient conditions for the optimality of (12), under the Assumptions 3.a, 3.b, and 3.d. Since there is a unique primal optimal solution, denoted by $z^*(x)$, for (12), there is at least one KKT solution $(z^*(x), \lambda^*)$ which satisfies the KKT conditions (14). In particular, the KKT solution satisfies the zero gradient condition (14a), which can be expanded using (13) as

$$\underbrace{\begin{bmatrix} \nabla_z A^{iT} & 0_{l \times r^i} \\ 0_{l \times r^j} & \nabla_z A^{jT} \end{bmatrix} (x, z^*(x))}_{= \nabla_z A(x, z^*(x))^T} \begin{bmatrix} \lambda^{*iT} \\ \lambda^{*jT} \end{bmatrix} = \begin{bmatrix} -2s^*(x) \\ 2s^*(x) \end{bmatrix}. \tag{40}$$

If $\mathcal{C}^i(x^i) \cap \mathcal{C}^j(x^j) = \emptyset$, the minimum distance between the two sets is greater than zero. So, $z^{*i}(x) \neq z^{*j}(x)$ and $s^*(x) \neq 0$. From (40), since the RHS is not zero, $\lambda^{*i}_{k^i} > 0$ and $\lambda^{*j}_{k^j} > 0$ for some $k^i \in [r^i]$ and $k^j \in [r^j]$.

The Hessian of L with respect to z is,

$$\nabla^2_z L(x, z^*(x), \lambda^*) = \begin{bmatrix} 2I & -2I \\ -2I & 2I \end{bmatrix} + \tag{41}$$
$$\begin{bmatrix} \sum_{m=1}^{r^i} \lambda^{*i}_m \nabla^2_z A^i_m & 0 \\ 0 & \sum_{n=1}^{r^j} \lambda^{*j}_n \nabla^2_z A^j_n \end{bmatrix} (x, z^*(x)).$$

Using $\lambda^* \geq 0$ and the strong convexity Assumption 3.b,

$$\sum_{m=1}^{r^i} \lambda^{*i}_m \nabla^2_z A^i_m(x, z^*(x)) \succeq \lambda^{*i}_{k^i} \nabla^2_z A^i_{k^i}(x, z^*(x)) \succ 0.$$

So, the first term in (41) is positive semi-definite, and the second term is positive definite, meaning that $\nabla^2_z L(x, z^*(x), \lambda^*) \succ 0$. Therefore, (15) is satisfied, and SSOSC holds at x.

E. Proof of Thm. 2

We first prove two preliminary results to prove Thm. 2. The following lemma, which is similar to [51, Lem. 1], shows that the set of constraints (23e)-(23g) for any state $x \in \mathcal{X}$ has fewer constraints than for a point in its vicinity. We show this using the index sets $\mathcal{J}_0, \mathcal{J}_1$, and $\mathcal{J}_2$.

Lemma 11. *Consider a sequence $\{x_{(p)}\} \to x$. Then, $\exists P \in \mathbb{N}$ such that $\forall p \geq P$,*

$$\mathcal{J}_1(x) \subseteq \mathcal{J}_1(x_{(p)}), \tag{42a}$$
$$\mathcal{J}_0(x)^c \subseteq \mathcal{J}_0(x_{(p)})^c, \tag{42b}$$
$$\mathcal{J}_2(x) \subseteq \mathcal{J}_{2,\epsilon}(x) \subseteq \mathcal{J}_{2,\epsilon}(x_{(p)}). \tag{42c}$$

Proof. First, we conclude a fact about continuous functions. Let $\gamma : \mathcal{X} \to \mathbb{R}$ be a continuous function, and let $\gamma(x) = \epsilon_1 < \epsilon$. By continuity of γ, $\lim_{p\to\infty} \gamma(x_{(p)}) = \gamma(x)$. Choosing $\epsilon_2 < \epsilon - \epsilon_1$, we obtain $P \in \mathbb{N}$ such that $\forall p \geq P$,

$$|\gamma(x_{(p)}) - \gamma(x)| = |\gamma(x_{(p)}) - \epsilon_1| < \epsilon_2 < \epsilon - \epsilon_1.$$

So, $\forall p \geq P$, $\gamma(x_{(p)}) < \epsilon$.

The set of indices $\mathcal{J}_1$ is such that $\lambda^*_k > 0 \ \forall k \in \mathcal{J}_1$. Since λ^* is continuous, (42a) follows. Similarly, $\mathcal{J}_0^c \ni k$ is the set of indices such that $A_k(z^*) < 0$, and (42b) also holds, by continuity of z^* and A_k. $\mathcal{J}_2(x) \subseteq \mathcal{J}_{2,\epsilon}(x) \ \forall \epsilon > 0$ by definition and, similar to (42a) and (42b), (42c) is true because of continuity of λ^*, z^*, and A. ☐

Next, we prove some general results about optimization problems, adopted from [1, App. 1].

Lemma 12. *Let $\mathcal{O}$ be an optimization problem as: $\phi^*(\theta) = \max_v\{\phi(\theta, v) : v \in \Gamma(\theta)\}$, with $\Gamma(\theta) = \{v : \gamma_k(\theta, v) \leq 0, k \in \mathcal{J}(\theta)\}$, where $\mathcal{J}$ is some index set dependent on θ. Let ϕ and γ_k be continuous, and $\Gamma(\theta)$ be uniformly bounded and non-empty. Define a corresponding optimization problem $\bar{\mathcal{O}}$ as: $\bar{\phi}^*(\theta) = \max_v\{\phi(\theta, v) : v \in \bar{\Gamma}(\theta)\}$, with $\bar{\Gamma}(\theta) = \{v : \gamma_k(\theta, v) \leq 0, k \in \bar{\mathcal{J}}\}$, where $\bar{\mathcal{J}} \subseteq \mathcal{J}(\theta) \ \forall\theta$. Also let $\bar{\Gamma}(\theta)$ be uniformly bounded. Then,*

(a) $\bar{\phi}^*(\theta) \geq \phi^*(\theta) \ \forall\theta$.

(b) Let $\{\theta_{(p)}\} \to \theta$. Then, $\bar{\phi}^*(\theta) \geq \limsup_{p\to\infty} \bar{\phi}^*(\theta_{(p)})$.

Proof. Note that $\bar{\Gamma}(\theta) \neq \emptyset \ \forall\theta$ since $\bar{\Gamma}(\theta) \supseteq \Gamma(\theta) \neq \emptyset$, which holds because $\bar{\Gamma}(\theta)$ has fewer constraints than $\Gamma(\theta)$.

(a) The optimization problems $\mathcal{O}$ and $\bar{\mathcal{O}}$ share the same cost function, and $\bar{\Gamma}(\theta) \supseteq \Gamma(\theta)$. So, $\bar{\phi}^*(\theta) \geq \phi^*(\theta) \ \forall\theta$.

(b) $\bar{\Gamma}(\theta)$ is uniformly bounded and ϕ is a continuous function. So, the sequence $\{\bar{\phi}^*(\theta_{(p)})\}$ is bounded. Choose a subsequence $\{\bar{\phi}^*(\theta_{(p_k)})\}$ such that $\lim_{k\to\infty} \bar{\phi}^*(\theta_{(p_k)}) = \limsup_{p\to\infty} \bar{\phi}^*(\theta_{(p)})$. Since $\bar{\Gamma}(\theta)$ is closed (it is the preimage of a closed set under the continuous function $\gamma_{\bar{\mathcal{J}}}$) and bounded, we can choose a bounded sequence of optimal solutions $\{\sigma_{(p_k)}\}$ such that $\phi(\theta_{(p_k)}, \sigma_{(p_k)}) = \bar{\phi}^*(\theta_{(p_k)})$. $\{\sigma_{(p_k)}\}$ is a bounded sequence and hence we can select a converging subsequence, also represented by $\{\sigma_{(p_k)}\}$, such that $\{\sigma_{(p_k)}\} \to \sigma$. By continuity of γ_j,

$$\gamma_j(\theta, \sigma) = \lim_{k\to\infty} \gamma_j(\theta_{(p_k)}, \sigma_{(p_k)}) \leq 0 \quad \forall j \in \bar{\mathcal{J}},$$

i.e. $\sigma \in \bar{\Gamma}(\theta)$. Finally, by continuity of ϕ,

$$\bar{\phi}^*(\theta) \geq \phi(\theta, \sigma) = \lim_{k\to\infty} \phi(\theta_{(p_k)}, \sigma_{(p_k)}),$$
$$= \lim_{k\to\infty} \bar{\phi}^*(\theta_{(p_k)}) = \limsup_{p\to\infty} \bar{\phi}^*(\theta_{(p)}).$$ ☐

We can now prove the safety result, Thm. 2.

Proof of Theorem 2. Let $\mathcal{F}_u(x)$ be the feasible set of control inputs of (23) for a given x. By the feasibility assumption, $\mathcal{F}_u(x) \neq \emptyset$. $\mathcal{F}_u(x)$ represents the set of control inputs, at a given state x, that guarantees collision avoidance. Let u be a control law such that $u(x) \in \mathcal{F}_u(x) \ \forall x \in \mathcal{X}$. Since u can be a discontinuous function, the closed loop trajectory is computed, using the Filippov operator (4), as $\dot{x}^i(t) \in F[f^i + g^i u^i](x^i(t))$. Thus, for safety we need to ensure that $F[u](x) \subseteq \mathcal{F}_u(x)$, i.e. the set of control inputs obtained after using the Filippov operator still satisfies (23).

The overview of the proof is as follows: First, given any control law $u(x)$ satisfying $u(x) \subseteq \mathcal{F}_u(x) \ \forall x \in \mathcal{X}$, we show that $F[u](x) \subseteq \mathcal{F}_u(x) \ \forall x \in \mathcal{X}$. Next, we show that if $F[u](x) \subseteq \mathcal{F}_u(x)$, then safety is guaranteed, i.e. there is no collision between the robots.

Consider a measurable control law $u : \mathcal{X} \to \mathcal{U}$ such that $u(x) \in \mathcal{F}_u(x)$.

1) $F[u](x) \subseteq \mathcal{F}_u(x) \ \forall x \in \mathcal{X}$:

We use Property 2 to prove this. Consider sequences $\{x_{(p)}\} \to x$ and $\{u(x_{(p)})\} \to u$. If we can show that $u \in \mathcal{F}_u(x)$, then by Property 2, $F[u](x) \subseteq \mathcal{F}_u(x)$. Consider the following optimization problem, $\mathcal{O}(x, y, u)$:

$$g(x, y, u) = \underset{\{\mathring{z}, \mathring{\lambda}\}}{\operatorname{argmax}} \ 2(z^{*i}(x) - z^{*j}(x))(\mathring{z}^i - \mathring{z}^j) \tag{43a}$$
$$\text{s.t.} \quad \mathring{x} = f(x) + g(x)u, \tag{43b}$$
$$\nabla^2_z L\mathring{z} + \mathring{\lambda}\nabla_z A + \nabla_x \nabla_z L\mathring{x} = 0, \tag{43c}$$
$$\nabla_z A_k \mathring{z} = -\nabla_x A_k \mathring{x}, \quad k \in \mathcal{J}_1(y), \tag{43d}$$
$$\nabla_z A_k \mathring{z} \leq -\nabla_x A_k \mathring{x}, \quad k \in \mathcal{J}_{2,\epsilon}(y),$$
$$\mathring{\lambda}_k = 0, \ k \in \mathcal{J}_0(y)^c, \quad \mathring{\lambda}_k \geq 0, \quad k \in \mathcal{J}_{2,\epsilon}(y), \tag{43e}$$
$$\mathring{\lambda}_k(\nabla_z A_k \mathring{z} + \nabla_x A_k \mathring{x}) = 0, \quad k \in \mathcal{J}_{2,\epsilon}(y), \tag{43f}$$
$$\|\mathring{z}\| \leq M, \quad \|\mathring{\lambda}\| \leq M, \tag{43g}$$
$$u \in \mathcal{U}. \tag{43h}$$

The cost, (43a), of (43) corresponds to the LHS of (23b), whereas the constraints (43b)-(43h) correspond to (23c)-(23i). Note that the variable y is used as an argument for the index sets $\mathcal{J}_0, \mathcal{J}_1$, and $\mathcal{J}_2$ in (43d)-(43f). Since $u(x_{(p)})$ is feasible at

$x_{(p)}$, we have that $g(x_{(p)}, x_{(p)}, u(x_{(p)})) \geq -\alpha \cdot h(x_{(p)})$. Note that u is feasible at x if $g(x,x,u) \geq -\alpha \cdot h(x)$, which is what we want to prove.

By Lem. 11, $\exists P \in \mathbb{N}$ such that $\forall p \geq P$, (42) holds. We truncate the sequences $\{x_{(p)}\}$ and $\{u(x_{(p)})\}$ for $p < P$, for convenience. Also note that the costs and constraints in (43) are continuous by Assumption 3.a, and continuity of z^* and λ^*. Since the optimization problem $\mathcal{O}(x_{(p)}, x_{(p)}, u(x_{(p)}))$ contains constraints with an index set, it can be related to the optimization problem $\mathcal{O}$ in Lem. 12. Similarly, by (42), $\mathcal{O}(x_{(p)}, x, u(x_{(p)}))$ can be related to $\bar{\mathcal{O}}$ in Lem. 12. The variable θ corresponds to (x,u) and v corresponds to $(\mathring{z}, \mathring{\lambda})$. Constraint (43g) ensures that $\Gamma(\theta)$ is uniformly bounded, while feasibility of (23) ensures non-empty $\Gamma(\theta)$. Using Lem. 12,

$$\begin{aligned} g(x,x,u) &\geq \limsup_{p\to\infty} g(x_{(p)}, x, u(x_{(p)})) \quad \text{(by Lem. 12.b)}, \\ &\geq \limsup_{p\to\infty} g(x_{(p)}, x_{(p)}, u(x_{(p)})) \quad \text{(by Lem. 12.a)}, \\ &\geq \limsup_{p\to\infty} -\alpha \cdot h(x_{(p)}) = -\alpha \cdot h(x). \end{aligned}$$

So, u is feasible at x, i.e. $u \in \mathcal{F}_u(x)$. By Property 2, $F[u](x) \subseteq \mathcal{F}_u(x) \; \forall x \in \mathcal{X}$.

2) The NCBF constraint (7) is satisfied at all times:

Let $u(x) \in \mathcal{F}_u(x)$ be any measurable feedback control law obtained using (23). The closed loop trajectory is obtained as a Filippov solution $x(t)$ for $t \in [0,T]$, satisfying $\dot{x}(t) \in F[f + gu](x(t))$ for almost all t. By Property 1, $F[f+gu](x(t)) = f(x(t)) + g(x(t))F[u](x(t))$. Thus, the Filippov solution $x(t)$ is such that $\dot{x}(t) = f(x(t)) + g(x(t))\bar{u}$ for almost all t, where $\bar{u} \in \mathcal{F}_u(x(t))$, meaning $\exists(\mathring{z}, \mathring{\lambda})$ such that $(\bar{u}, \mathring{z}, \mathring{\lambda})$ is feasible for (23).

Note that, by (42c), if $(\mathring{z}, \mathring{\lambda})$ satisfies (23d)-(23g) then it also satisfies (21). By Thm. 1, $\mathring{z}$ is the right derivative of z^* in the direction $\mathring{x}$ which, by (23c), equals $\dot{x}(t)$. Using (19) and (23b), and noting that $h(t)$ is absolutely continuous (see above Lem. 1),

$$\dot{h}(t) = 2(z^{*i}(t) - z^{*j}(t))(\mathring{z}^i - \mathring{z}^j) \geq -\alpha \cdot h(t),$$

for almost all $t \in [0,T]$. By Lem. 1, since $x(0) \in \mathcal{S}$, $h(t) > 0 \; \forall t \in [0,T]$, i.e. safety is maintained. □

Proofs for Polytopic Sets

F. Proof of Lem. 9

We first prove that a unique dual optimal solution exists to (26), then show that it is differentiable almost always. We also simplify the notation $A^i_{\mathcal{J}^i_0(t)}(t)$ to $A^i_{\mathcal{J}^i_0}(t)$

1) Uniqueness of the dual optimal solution: Since an optimal solution to (25) always exists, an optimal solution to (26) also exists. For any $t \in [0,T]$, choose $\bar{z}^* \in \mathcal{Z}^*(t)$ such that $\mathcal{J}^i(x^i(t), \bar{z}^{*i}) = \mathcal{J}^i(t)$ and $\mathcal{J}^j(x^j(t), \bar{z}^{*j}) = \mathcal{J}^j(t)$. By definition of $\bar{z}^*$ and $\mathcal{J}(t)$,

$$\begin{aligned} A^i_{[r^i]\setminus\mathcal{J}^i}(t)\bar{z}^{*i} &< b^i_{[r^i]\setminus\mathcal{J}^i}(t), \\ A^j_{[r^j]\setminus\mathcal{J}^j}(t)\bar{z}^{*j} &< b^j_{[r^j]\setminus\mathcal{J}^j}(t). \end{aligned} \tag{44}$$

For $\bar{z}^*$ to be an optimal solution, there must exist a $\bar{\lambda}^*$ satisfying the KKT conditions (28). From (28b) and (44),

$$\bar{\lambda}^{*i}_{[r^i]\setminus\mathcal{J}^i(t)} = 0, \quad \bar{\lambda}^{*j}_{[r^j]\setminus\mathcal{J}^j(t)} = 0. \tag{45}$$

The remaining components of $\bar{\lambda}^*$ must satisfy, from (28a),

$$\bar{\lambda}^{*i}_{\mathcal{J}^i(t)} A^i_{\mathcal{J}^i}(t) = -2s^{*T}(t), \;\; \bar{\lambda}^{*j}_{\mathcal{J}^j(t)} A^j_{\mathcal{J}^j}(t) = 2s^{*T}(t). \tag{46}$$

From the linear independence of $\mathcal{C}^i(x^i(t))$ and $\mathcal{C}^j(x^j(t))$ (Assumption 4.b), both $A^i_{\mathcal{J}^i}(t)$ and $A^i_{\mathcal{J}^i}(t)$ have full row-rank, and thus can be inverted. The non-zero components of the dual optimal solution $\bar{\lambda}^*$ can be computed, using (46), as

$$\bar{\lambda}^{*i}_{\mathcal{J}^i(t)} = -2s^{*T}(t)A^{i\dagger}_{\mathcal{J}^i}(t), \;\; \bar{\lambda}^{*j}_{\mathcal{J}^j(t)} = 2s^{*T}(t)A^{j\dagger}_{\mathcal{J}^j}(t), \tag{47}$$

where $(\cdot)^\dagger$ represents pseudo-inverse. The zero components of the optimal dual solution from (45) can be appended to (47) to obtain $\bar{\lambda}^*$. Note that $\bar{\lambda}^*$ is the unique solution corresponding to the primal optimal solution $\bar{z}^*$. Since $\bar{z}^*$ has the least number of active constraints among all primal solutions in $\mathcal{Z}^*(t)$, $\bar{\lambda}^*$ must be the unique solution of (26), corresponding to all primal solutions. We denote $\bar{\lambda}^*$ simply as $\lambda^*(t)$.

2) Right-differentiability of the dual optimal solution: Let t be a time when Assumption 5 holds, i.e., $\text{Aff}^i(\tau)$ and $\text{Aff}^j(\tau)$ have constant dimension for $\tau \in [t, t+\epsilon)$. Using $\mathcal{J}^i(x^i(\tau), \bar{z}^{*i}(\tau)) = \mathcal{J}^i_0(\tau)$ and Assumption 4.b, we can see that $A^i_{\mathcal{J}^i_0}(\tau)$ has full row-rank for $\tau \in [t, t+\epsilon)$ (and similarly for j). The dimension of the affine space $\text{Aff}^i(\tau)$ is equal to the nullity of $A^i_{\mathcal{J}^i_0}(\tau)$. Thus, the constant dimension of $\text{Aff}^i(\tau)$ implies that $A^i_{\mathcal{J}^i_0}(\tau)$ has a constant rank, which, by linear independence of its rows, means that $A^i_{\mathcal{J}^i_0}(\tau)$ has a constant number of rows ($= |\mathcal{J}^i_0(\tau)|$). Since $\text{Aff}^i(\tau)$ and $\text{Aff}^j(\tau)$ have constant dimensions for $\tau \in [t, t+\epsilon)$ they can be represented equivalently as

$$\begin{aligned} \text{Aff}^i(\tau) &:= \{\tilde{z}^i(\tau) + N^i(\tau)y^i : y^i \in \mathbb{R}^{|\mathcal{J}^i_0(\tau)|}\}, \\ \text{Aff}^j(\tau) &:= \{\tilde{z}^j(\tau) + N^j(\tau)y^j : y^j \in \mathbb{R}^{|\mathcal{J}^j_0(\tau)|}\}, \end{aligned} \tag{48}$$

where $\tilde{z}^i(\tau)$ ($\tilde{z}^j(\tau)$) is some point on $\text{Aff}^i(\tau)$ ($\text{Aff}^j(\tau)$), and $N^i(\tau)$ ($N^j(\tau)$) represents basis vectors on $\text{Aff}^i(\tau)$ ($\text{Aff}^j(\tau)$), for $\tau \in [t, t+\epsilon_1)$ and $\epsilon_1 \in (0, \epsilon)$. By Assumption 4.a and Assumption 5, $\tilde{z}^i(\tau)$, $\tilde{z}^j(\tau)$, $N^i(\tau)$, and $N^j(\tau)$ can be chosen to be continuously differentiable for $\tau \in [t, t+\epsilon_1)$. Also Assumption 5 implies that $N^i(\tau)$, $N^j(\tau)$ have constant ranks, and $[-N^i(\tau), N^j(\tau)]^T$ has a constant nullity for $\tau \in [t, t+\epsilon_1)$, which additionally implies that $[-N^i(\tau), N^j(\tau)] = N(\tau)$ has constant rank for $\tau \in [t, t+\epsilon_1)$.

The separating vector $s^*(\tau)$ is the unique vector from $\text{Aff}^i(\tau)$ to $\text{Aff}^j(\tau)$ that is perpendicular to both of them. These three constraints (that define $s^*(\tau)$ and establish orthogonality of $s^*(\tau)$ to $\text{Aff}^i(\tau)$ and $\text{Aff}^j(\tau)$) can be written in the form of a system of linear equalities as,

$$\begin{aligned} &\tilde{z}^i(\tau) + N^i(\tau)y^i - (\tilde{z}^j(\tau) + N^j(\tau)y^j) = s, \\ &-N^{iT}(\tau)s = 0, \qquad N^{jT}(\tau)s = 0, \end{aligned}$$

which can be written in a matrix form as,

$$\begin{bmatrix} I & -N^i(\tau) & N^j(\tau) \\ -N^{iT}(\tau) & 0 & 0 \\ N^{jT}(\tau) & 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ y^i \\ y^j \end{bmatrix} = \begin{bmatrix} \tilde{z}^i(\tau) - \tilde{z}^j(\tau) \\ 0 \\ 0 \end{bmatrix}. \tag{49}$$

The LHS matrix has constant rank for $\tau \in [t, t+\epsilon_1)$, since $\text{rank}(N(\tau)) = \text{rank}(N^T(\tau)N(\tau))$. By Lem. 3, at least one solution exists to (49). Therefore, by inverting a suitable sub-matrix, we can find a continuously differentiable vector $[\bar{s}^T(\tau), \bar{y}^{iT}(\tau), \bar{y}^{jT}(\tau)]^T$ that solves (49). Moreover, $s^*(\tau)$ is the unique vector that solves (49). Next, we can solve for the dual optimal solutions using (47). By the continuous differentiability of A^i, A^j, and s^* in the interval $[t, t+\epsilon_1)$, the dual optimal solutions are continuously differentiable in $[t, t+\epsilon_1)$. Thus, the dual optimal solution $\lambda^*(t)$ is continuous and right-differentiable for almost all $t \in [0, T]$.

G. Proof of Lem. 10

Let $(\dot{\lambda}^i, \dot{\lambda}^j)$ be any feasible solution to (31) for some $t \in [0, T]$. We want to integrate the constraints of (31) to find dual solutions $(\bar{\lambda}^i(\tau), \bar{\lambda}^j(\tau))$, $\tau \in [t, t+\epsilon)$, which satisfy the feasibility constraints of the dual QP (26), i.e.

$$(\bar{\lambda}^i(t), \bar{\lambda}^j(t)) = (\lambda^{*i}(t), \lambda^{*j}(t)), \quad (50)$$
$$(\bar{\lambda}^i(\tau), \bar{\lambda}^j(\tau)) \text{ is dual feasible for (26)},$$
$$\frac{d}{d\tau}(\bar{\lambda}^i, \bar{\lambda}^j)\bigg|_{\tau=t_+} = (\dot{\lambda}^i, \dot{\lambda}^j).$$

In order to find a solution of (50), we first define

$$\bar{\lambda}^i(\tau) = \lambda^{*i}(t) + (\tau - t)\dot{\lambda}^i, \quad (51)$$
$$\bar{\lambda}^j(\tau) = \lambda^{*j}(t) + (\tau - t)\dot{\lambda}^j + e(\tau), \quad (52)$$

where $\tau \in [t, t+\epsilon)$, for $\epsilon > 0$ such that $\bar{\lambda}^i(\tau), \bar{\lambda}^j(\tau) - e(\tau) \geq 0,\ \forall \tau \in [t, t+\epsilon)$. $e(\tau)$ will be chosen to satisfy the feasibility constraints of the dual QP (26). Let

$$\bar{s}(\tau) = \bar{\lambda}^i(\tau) A^i(\tau) + (\bar{\lambda}^j(\tau) - e(\tau)) A^j(\tau).$$

We can think of $\bar{s}(\tau)$ as a separating vector between polytopes $\mathcal{C}^i(\tau)$ and $\mathcal{C}^j(\tau)$. Note that $\bar{s}(\tau)$ is a continuously differentiable function of τ. Next we show that $e(\tau)$ can be chosen such that $(\bar{\lambda}^i(\tau), \bar{\lambda}^j(\tau))$ satisfies (50). By Lem. 9, $e(\tau)$ can be chosen in an optimal manner, such that it satisfies

$$e(\tau) A^j(\tau) = -\bar{s}(\tau), \quad e(\tau) \geq 0, \quad e(t) = 0.$$

Additionally, by considering different invertible submatrices of $A^j(\tau)$ and by continuous differentiability of $\bar{s}(\tau)$, $e(\tau)$ is right-differentiable at $\tau = t$ and $\dot{e}(t_+) = 0$ (see [50, Thm. 3]). Using (51) and the above definition of $e(\tau)$, $(\bar{\lambda}^i(\tau), \bar{\lambda}^j(\tau))$ satisfies (50).

So, $(\bar{\lambda}^i(\tau), \bar{\lambda}^j(\tau))$ is a dual feasible solution and has cost less than $h(\tau)$, i.e. $\Lambda(\tau, \bar{\lambda}^i(\tau), \bar{\lambda}^j(\tau)) \leq h(\tau),\ \forall \tau \in [t, t+\epsilon)$, and $\Lambda(t, \bar{\lambda}^i(t), \bar{\lambda}^j(t)) = h(t)$. Differentiating the cost yields,

$$\dot{h}(t) \geq \dot{\Lambda}(t, \lambda^{*i}(t), \lambda^{*j}(t), \dot{\lambda}^i, \dot{\lambda}^j),$$

for all feasible $(\dot{\lambda}^i, \dot{\lambda}^j)$, and thus $\dot{h}(t) \geq g(t)$ for almost all $t \in [0, T]$.

H. Proof of Thm. 3

Let $\mathcal{F}_u(x)$ be the feasible set of control inputs of (35) for a given x. The overview of the proof is similar to that in Thm. 2: First, given any control law $u(x)$ satisfying $u(x) \in \mathcal{F}_u(x)\ \forall x \in \mathcal{X}$, we show that $F[u](x) \subseteq \mathcal{F}_u(x)\ \forall x \in \mathcal{X}$. Next, we show that if $F[u](x) \subseteq \mathcal{F}_u(x)$, then safety is guaranteed. Similar to (43), the following optimization problem is used to prove the first part:

$$\begin{aligned}
g(x, y, u) = &\max_{\{\dot{\lambda}\}} \dot{\Lambda}(t, \lambda^*(x), \dot{\lambda}, u) \\
\text{s.t.} \quad & \dot{\lambda}^i A^i(x) + \lambda^{*i}(x)(\mathcal{L}_{f^i} A^i(x) + \mathcal{L}_{g^i} A^i(x) u) \\
& \quad + \lambda^{*j}(x)(\mathcal{L}_{f^j} A^j(x) + \mathcal{L}_{g^j} A^j(x) u) = -\dot{\lambda}^j A^j(x) \\
& \dot{\lambda}^i_k \geq 0 \ \text{ if } \ \lambda^{*i}_k(y) < \epsilon, \qquad \dot{\lambda}^j_k \geq 0 \ \text{ if } \ \lambda^{*j}_k(y) < \epsilon, \\
& |\dot{\lambda}^i| \leq M, |\dot{\lambda}^j| \leq M.
\end{aligned}$$

The rest of the proof is the same as that of Thm. 2 and has been omitted for brevity.

Acknowledgement

The authors gratefully acknowledge Dr. Somayeh Sojoudi at the University of California, Berkeley for her valuable comments on the properties of parametric optimization problems.

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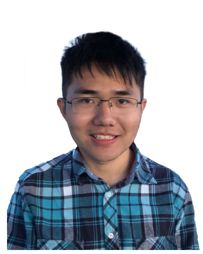

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