Multiple quadrotors carrying a flexible hose: dynamics, differential flatness and control

Prasanth Kotaru, Koushil Sreenath

Abstract: Using quadrotors UAVs for cooperative payload transportation using cables has been actively gaining interest in the recent years. Understanding the dynamics of these complex multi-agent systems would help towards designing safe and reliable systems. In this work, we study one such multi-agent system comprising of multiple quadrotors transporting a flexible hose. We model the hose as a series of smaller discrete links and derive a generalized coordinate-free dynamics for the same. We show that certain configurations of this under-actuated system are differentially-flat. We linearize the dynamics using variation-based linearization and present a linear time-varying LQR to track desired trajectories. Finally, we present numerical simulations to validate the dynamics, flatness and control.

Keywords: Coordinate-free dynamics, variation based linearization, co-operative control, aerial manipulation, differential-flatness

1. INTRODUCTION

Aerial manipulation is active research for many years now, due to the simplicity of the dynamics and control of multi-rotors. The ubiquity of these aerial vehicles resulted in using these UAVs in a wide range of applications. Few such applications include search and rescue [Bernard et al. (2011)] and disaster management like using UAVs to monitor forest fires [Merino et al. (2012)]. Payload delivery using aerial vehicles [X-Wing (2019), PrimeAir (2019), Palunko et al. (2012)] is another application that earned much attention in the last few years.

One extension of the payload carrying research is developing multi-rotor vehicles for active fire-fighting [Aerones (2018)] using tethered hoses that carries water and power. This enables carrying a fire hose to heights higher than a typical fire-truck ladder as well as longer flight times due to the tethered power supply. Multi-rotors are also being used to help string power cables between poles [SkyScopes (2017)], which typically is achieved using manned helicopters. To achieve a stable and safe control for these complex systems, it is important to understand the underlying governing principles and dynamics. In this work, we aim to model and control the dynamics of a multiple quadrotor system carrying a flexible cable/hose.

1.1 Related Work

There is a lot of literature on co-operative aerial manipulation, especially towards grasping and transporting payloads using multiple quadrotors [Maza et al. (2009), Mellinger et al. (2013), Jiang and Kumar (2012), Lee and Kim (2017), Michael et al. (2011)]. Trajectory tracking control control for point-mass/rigid-body payloads suspended from multiple quadrotors is studied in [Lee et al. (2013), Goodarzi and Lee (2016), Sreenath and Kumar (2013), Wu and Sreenath (2014)]. Similarly, for loads suspended using flexible cables, stabilizing controllers are presented in [Goodarzi et al. (2015), Goodarzi and Lee (2015)] and these systems are shown differentially-flat in [Kotaru et al. (2018)]. Tethered aerial vehicles have also been extensively studied in the literature, for instance stabilization of tethered quadrotor and nonlinear-observers for the same are discussed in [Lupashin and D’Andrea (2013), Nicotra et al. (2014), Tognon and Franchi (2015)]. Geometric control of a tethered quadrotor with a flexible tether is presented in [Lee (2015)].
Most of the work discussed in the previous section models the tethers/cables either as rigid-links or as a series of links. Partial differential equations are used to model a continuous mass system, such as the aerial refueling cable shown in [Liu et al. (2017)]. However, modeling the aerial cable as a finite-segment lumped mass [Williams and Trivalo (2007), Ro and Kamman (2010)] is quite common in the literature due to the finite dimensionality of the state-space. However, most of these works assume orientation angles in the local frame to represent the attitude of the links. This results in complex equations of motion for the system that are also prone to singularities in case of aggressive motions. Therefore, in this work, we make use of coordinate-free representation that results in singularity-free and compact equations of motion.

1.2 Challenges

Multiple quadrotors carrying a flexible hose has multiple challenges in both modeling the dynamics and also designing a controller. Even though modeling the hose as a finite-segment lumped mass results in a finite-dimensions state space, it would still result in a higher number of states depending on the choice of the number of discrete links. In addition, developing a controller is challenging due to the high under-actuation in the system. If the state of the cable is not taken into consideration, the vibrations in the cable will result in instability.

1.3 Contributions

In this paper, we build upon the work done in the literature to develop the dynamics and control of multiple quadrotors carrying a flexible hose. The contributions of this work are as follows,

- We derive a generalized coordinate-free dynamics for multiple quadrotors carrying a flexible hose system. These dynamics can be extended to a tethered multiple quadrotor system.
- We show that this system is differentially-flat for certain configurations.
- We present variation-based linearized dynamics and implement a time-varying LQR to track a time-varying desired trajectory.
- Finally, we present numerical simulations to validate the dynamics and control.

To the best of authors knowledge, this is novel configuration of multiple quadrotors with a flexible hose and has not been studied prior to this work.

1.4 Organization

Rest of the paper is organized as follows. Section 2 explains the system definition, notations and presents the derivation of the dynamics. In Section 3 we show that the system is differentially-flat. In section 4, we present a LQR control on the variation-linearized dynamics. Section 5 presents numerical simulations validating the proposed controller. Finally, some of the limitations in this paper and potential directions to address them are discussed in Section 6. Concluding remarks are in Section 7.

2. DYNAMICS

Consider a flexible hose connected to multiple quadrotor UAVs as shown in Figure 1. In this section, we present the coordinate-free dynamics for this system. We consider the following assumptions before proceeding to derive the dynamics:

a1. No water/water-flow in the hose and thus also no pressure forces;

a2. Hose is modeled as a series of n smaller links with lumped-mass at the end connected by spherical joints;

a3. We also ignore the hose mechanical properties like stiffness, torsional forces etc, and instead assume each link is mass-less with lumped-point-masses at the ends; and

a4. We also assume that the quadrotors attach to the hose at their respective center-of-masses.

In the following section, we present the notation used to describe the system.

2.1 Notation

Dynamics for the model are defined using geometric-representation of the states. Each link is a spherical-joint and is represented using a unit-vector $\mathbf{r}_i \in \mathbb{R}^3$, rotation matrix $\mathbf{R} \in SO(3)$, and its mass $m_i$ and inertia $I_i$. Let the set $I_j \in \{1, \ldots, n\}$ be the number of quadrotors. $R_j \in SO(3)$ is the quadrotor attitude, $m_{Q_j}, J_j$ is its mass and inertia matrix and $f_j \in \mathbb{R}, M_j \in \mathbb{R}^3$ is the corresponding thrust and moment for all $j \in \mathcal{I}$. Finally, the configuration space of this system is given as $Q := \mathbb{R}^3 \times (S^2)^n \times (SO(3))^n$. Table 1 lists the various symbols used in this paper.

2.2 Derivation

The kinematic relation between the different link positions is given using link attitudes as,

$$x_i = x_0 + \sum_{k=1}^{i} l_k q_k, \quad \forall i \in S \setminus \{0\},$$

and the corresponding velocities and accelerations are related as,

$$v_i = v_0 + \sum_{k=1}^{i-1} l_k \dot{q}_k, \quad a_i = a_0 + \sum_{k=1}^{i-1} l_k \ddot{q}_k.$$

Potential energy $U : TQ \to \mathbb{R}$ of the system, due to hose and quadrotors’ mass is computed as shown below,

$$U = \sum_{i \in \mathcal{S}} m_i x_i \cdot \mathbf{g}_3,$$

where $m_i = m_i + m_{Q_i} l_i$ is the net-mass at index $i$ and $1_i := 1_{\mathcal{I}}(i) = \begin{cases} 1 & \text{if } i \in \mathcal{I} \\ 0 & \text{else} \end{cases}$ is an indicator function for the set $\mathcal{I}$. Kinetic energy $T : TQ \to \mathbb{R}$ is similarly given as,

$$T = \sum_{i \in \mathcal{S}} \frac{1}{2} m_i \langle v_i, v_i \rangle + \sum_{i \in \mathcal{S}} \frac{1}{2} \langle \Omega_j, J_j \Omega_j \rangle,$$

where $\Omega_j$ is the angular velocity of the quadrotor $j$ in its body-frame. Dynamics of the system are derived using the Lagrangian method, where Lagrangian $\mathcal{L} : TQ \to \mathbb{R}$, is given as,

$$\mathcal{L} = T - U.$$
We derive the equations of motion using the Langrange-d’Alembert’s principle of least action, given below,
\[ \delta \int_{t_0}^{t_f} L \, dt + \int_{t_0}^{t_f} \delta W_c \, dt = 0, \tag{8} \]
where \( \delta W_c \) is the infinitesimal work done by the external forces. \( \delta W_c \) can be calculated as,
\[ \delta W_c = \sum_{j \in \mathcal{E}} \left( (W_{1,j}, \dot{M}_j) + (W_{2,j}, f_R \delta e_3) \right), \tag{9} \]
and
\[ W_{1,j} = R_j T^e \delta R_j, \tag{10} \]
\[ W_{2,j} = \delta x_j = \delta x_0 + \sum_{k=1}^{l_j} l_k \delta q_k, \tag{11} \]
are variational vector fields [Goodarzi et al. (2015)] corresponding to quadrotor attitudes and positions. The infinitesimal variations on \( q \) and \( R \) are expressed as,
\[ \delta q = \xi^q \cdot q = -q^\xi \cdot \xi, \xi \in \mathbb{R}^3 \text{ s.t. } \xi \cdot q = 0, \]
\[ \delta R = R_0 \Omega^\xi, \delta \Omega^\xi = (\Omega^\eta)^\xi + \eta^\xi, \eta \in \mathbb{R}^3, \]
where \( \Omega = q \cdot \Omega \). The cross-map is defined as \((\cdot)^\xi : \mathbb{R}^3 \to \mathfrak{so}(3)\) for \( x, y \in \mathbb{R}^3 \). Similarly, variations on the link positions are given as,
\[ \delta x_i = \delta x_0 + \sum_{k=1}^{l_i} l_k \delta q_k = \delta x_0 - \sum_{k=1}^{l_i} l_k q_k^\xi \cdot \xi_k, \tag{12} \]
\[ \delta v_i = \delta v_0 + \sum_{k=1}^{l_i} k \delta q_k = \delta v_0 - \sum_{k=1}^{l_i} k (q_k^\xi \cdot \xi_k + q_k^\xi \cdot \xi_k). \tag{13} \]
Finally, we obtain the equations of motion for the system by solving (8). See Appendix A for the detailed derivation. Equations of motion for the multiple quadrotors carrying a flexible hose are given in (1)-(3). Note the mass matrix \( M_{(q)} \) is the function of link attitudes \( q \) \( = \{ q_1, q_2, \ldots, q_n \} \) and we use the following notation similar to [Goodarzi et al. (2014)]
\[ M_{00} = \sum_{k=0}^{n} m_k, \quad M_{0i} = l_i \sum_{k=1}^{n} m_k, \]
\[ M_{ij} = M_{0i} = M_{0j}, \quad M_{ijk} = \sum_{k=\max(i,j)}^{n} m_k l_k l_j. \tag{14} \]

Remark 1. In (2), note the use of \( f_i, R_i \) for \( i \notin \mathcal{I} \), (since \( i \notin \mathcal{I} \) implies no quadrotor is attached at index \( i \) and thus cannot have \( f_i \) and \( R_i \)), however, this is done for convenience. \( i \notin \mathcal{I} \implies f_i = 0 \) and thus \( f_i R_i \delta e_3 = 0 \), there by ensuring the right inputs to the system.

Remark 2. Degrees of freedom for the multiple quadrotors carrying a flexible hose is \( DOF = 3(nQ+1)+2n \) where \( 2n \) corresponds to the link attitudes DOF, \( 3nQ \) the rotational DOF of the quadrotors and \( 3 \) for the initial position \( x_0 \). Similarly, the degrees of actuation is \( DOA = 4nQ \) corresponding to the 4 inputs for each quadrotor. Therefore, the degrees of under-actuation is computed as \( DOU \times A = 2n + 3 - nQ \). For a typical setup we have \( n >> nQ \), i.e., system is highly under-actuated.

Remark 3. For a tethered system, without loss of generality, we can assume \( x_0 \equiv 0 \forall t \), i.e., the system is tethered to origin of the inertial frame and derive the dynamics as previous. Equations of motion for this system would be same as (1)-(3), without the equation corresponding to \( \dot{v}_0 \).

3. DIFFERENTIAL FLATNESS

In the previous section, we derived the dynamics for multiple quadrotors carrying a flexible hose. The system is under-actuated and thus the control of the system is challenging. In this section, we show that the system is differentially-flat. A nonlinear system is differentially-flat if a set of outputs of the systems (equal to the number of inputs) and their derivatives can be used to determine the states and inputs without integration.

Definition 1. Differentially-Flat System, [Murray et al. (1995)]. A system \( \dot{x} = f(x, u) \in \mathbb{R}^n, u \in \mathbb{R}^m \), is differentially flat if there exists flat outputs \( y \in \mathbb{R}^m \) of the form \( y = y(x, u, \ldots, u^{(p)}) \) such that the states and the inputs can be expressed as \( x = x(y, \dot{y}, \ldots, y^{(q)}) \), \( u = u(y, \dot{y}, \ldots, y^{(q)}) \), where \( p, q \) are nonnegative finite integers.

A quadrotor is a differentially-flat system with the quadrotor center-of-mass and yaw as the flat outputs [Mellinger and Kumar (2011)]. A quadrotor with cable suspended load (with the cable modeled as a mass-less link) is also shown to be differentially-flat with load position and quadrotor yaw as the flat outputs [Sreenath et al. (2013)]. Similarly, a quadrotor with flexible cable suspended load,
with the cable modeled as series of smaller links is shown to be differentially-flat [Kotaru et al. (2018)]. Again, here load position and quadrotor yaw are the flat-outputs. For the system defined in this work, the quadrotor-flexible cable segments are connected in series. Unlike previous work where each quadrotor has only one segment connected to it, each quadrotor in this system can have 0, 1, 2

\[ S = \{0, 1, \ldots, n\} \]

\[ j \in I \quad \text{set of quadrotor indices} \]

\[ l_i \in \mathbb{R}^+ \]

\[ m_k \in \mathbb{R}^+ \]

\[ q_i \in S^2 \]

\[ \omega_i \in T_{q_i}S^2 \]

\[ I \subset S \]

\[ |I| = nQ \]

\[ x_{Qj} \equiv x_j \]

\[ R_j \in SO(3) \]

\[ \Omega_j \in T_{R_j}SO(3) \]

\[ m_{Qj}, J_j \]

\[ f_j \in \mathbb{R}, M_j \in \mathbb{R}^3 \]

\[ 1_i := 1_{\mathcal{I}(i)} = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{else} \end{cases} \]

\[ m_k = m_k + m_{Qj} \]

\[ u_k = (-\overline{m}_k \mathbf{e}_3 + f_k R_k \mathbf{e}_3 \mathbf{1}_k) \]

\[ \mathcal{I} \subset S \]

\[ |\mathcal{I}| = nQ \]

\[ x_{Qj} \equiv x_j \]

\[ R_j \in SO(3) \]

\[ \Omega_j \in T_{R_j}SO(3) \]

\[ m_{Qj}, J_j \]

\[ f_j \in \mathbb{R}, M_j \in \mathbb{R}^3 \]

\[ 1_i := 1_{\mathcal{I}(i)} = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{else} \end{cases} \]

\[ m_k = m_k + m_{Qj} \]

\[ u_k = (-\overline{m}_k \mathbf{e}_3 + f_k R_k \mathbf{e}_3 \mathbf{1}_k) \]

Fig. 2. Configuration of the multiple quadrotors carrying a flexible hose illustrating the differential-flatness and its flat-outputs (shown in red).

Table 1. List of various symbols used in this work. Note: \( k \in \mathcal{I}, j \in \mathcal{I}, I \subset \mathcal{I}, W F - \text{World frame}, BF - \text{Body-frame}, | \cdot | \) represents cardinality of a set.

<table>
<thead>
<tr>
<th>Variables</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \in \mathbb{R}^+ )</td>
<td>Number of links in the hose.</td>
</tr>
<tr>
<td>( S = {0, 1, \ldots, n} )</td>
<td>Set containing indices of the hose-segments.</td>
</tr>
<tr>
<td>( x_k \in \mathbb{R}^3 )</td>
<td>Position of the ( k^{th} ) point-mass of the hose in WF.</td>
</tr>
<tr>
<td>( v_k \in \mathbb{R}^3 )</td>
<td>Velocity of the ( k^{th} ) point-mass of the hose in WF.</td>
</tr>
<tr>
<td>( l_i \in \mathbb{R}^+ )</td>
<td>Length of the ( i^{th} ) hose segments.</td>
</tr>
<tr>
<td>( m_k \in \mathbb{R}^+ )</td>
<td>Mass of the ( k^{th} ) point-mass in the hose-segments.</td>
</tr>
<tr>
<td>( q_i \in S^2 )</td>
<td>Orientation of the ( i^{th} ) hose segment in the world frame.</td>
</tr>
<tr>
<td>( \omega_i \in T_{q_i}S^2 )</td>
<td>Angular velocity of the ( i^{th} ) hose segment in the world frame.</td>
</tr>
<tr>
<td>( I \subset S )</td>
<td>Set of indices where the hose is attached to the quadrotor.</td>
</tr>
<tr>
<td>(</td>
<td>I</td>
</tr>
<tr>
<td>( x_{Qj} \equiv x_j )</td>
<td>Center-of-mass position of the ( j^{th} ) quadrotor in the world-frame.</td>
</tr>
<tr>
<td>( R_j \in SO(3) )</td>
<td>Attitude of the ( j^{th} ) quadrotor w.r.t. world inertial frame.</td>
</tr>
<tr>
<td>( \Omega_j \in T_{R_j}SO(3) )</td>
<td>Angular velocity of the body w.r.t. WF in BF.</td>
</tr>
<tr>
<td>( m_{Qj}, J_j )</td>
<td>Mass &amp; inertia of the ( j^{th} ) quadrotor.</td>
</tr>
<tr>
<td>( f_j \in \mathbb{R}, M_j \in \mathbb{R}^3 )</td>
<td>Thrust and moment of the ( j^{th} ) quadrotor in the BF.</td>
</tr>
</tbody>
</table>

Differential-flatness is used in planning the system trajectories, where the flat outputs are used to plan in the lower-dimension space and the corresponding desired states and inputs are computed using differential flatness. In the next section, we present the linearized dynamics about any desired time-varying trajectory and use a linear controller to track desired trajectories.
4. CONTROL

Having presented differential-flatness in the previous section, we proceed to present control to track desired-trajectories generated using the flat-outputs in this section. As presented in the Remark 2, the given system is highly underactuated and thus controlling the system is challenging. In this section, we present a way to control the system by linearizing the dynamics in (1)-(3) given a desired time-varying trajectory\(^1\) \((x_{d}(t), v_{d}(t), q_{d}(t), \omega_{d}(t), R_{j}(d)(t), \Omega_{j}(d)(t))\) and then implementing a linear controller.

4.1 Variation Based Linearization

In this sub-section, we present the coordinate-free linear dynamics, obtained through variation based linearization of the nonlinear dynamics in (1)-(3). We use the variation linearization techniques described in Wu and Sreenath (2015) to obtain the linear dynamics. The error state of the linear dynamics are given as,

\[
\delta x = [\delta x, \delta \xi_1, \ldots, \delta \xi_{n_{\eta}}, \delta v, \delta \omega_1, \ldots, \delta \omega_{n_{d}, n_{Q}}, \delta M_{i}, \ldots, \delta \Omega_{i,j}^{n_{Q}}]^\top, \tag{15}
\]

and the corresponding inputs as,

\[
\delta u = [\delta f_{j_1}, \delta f_{j_2}, \ldots, \delta f_{j_{n_{Q}}}, \delta M_{i,j_1}^n, \ldots, \delta M_{i,j}^{n_{Q}}]^\top, \tag{16}
\]

where \(j_1, j_2, \ldots, j_{n_{Q}}\) are elements of \(\mathcal{I}\) arranged in increasing order. The individual elements of the error state are computed as,

\[
\begin{align*}
\delta x &= x - x_d, \\
\delta v &= v - v_d, \\
\delta \xi_i &= q_{d, i}^x, \\
\delta \omega_i &= \omega_i - (-q_i^x)^2 \omega_{id}, \\
\eta_j &= \frac{1}{2}(R_{j_d}^T R_{j_d} - R_j^T R_j)^\top, \\
\delta \Omega_j &= \Omega_j - R_{j_d}^T R_{j_d} \Omega_{j_d}.
\end{align*}
\]

Finally, the linearized dynamics (See Appendix C for detailed derivation of the linearized dynamics) about a time-varying desired trajectory are given below,

\[
\delta \dot{x} = A \delta x + B \delta u, \tag{17}
\]

\[
C \delta x = 0, \tag{18}
\]

where,

\[
A = \begin{bmatrix}
0_{3,3} & 0_{3,3} & I_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} \\
0_{3,3} & \alpha & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} \\
M_{(v_d)} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} \\
0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} \\
0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} \\
0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3}
\end{bmatrix}, \tag{19}
\]

\[
B = \begin{bmatrix}
O_{3,3} & O_{3,3} & O_{3,3} & O_{3,3} & O_{3,3} & O_{3,3} & O_{3,3} \\
M_{(\delta v)} & M_{(\delta v)} & M_{(\delta v)} & M_{(\delta v)} & M_{(\delta v)} & M_{(\delta v)} & M_{(\delta v)}
\end{bmatrix}, \tag{20}
\]

\[
C = \begin{bmatrix}
0_{3,3} & c_1 & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} \\
0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3}
\end{bmatrix}, \tag{21}
\]

\[
F = \begin{bmatrix}
O_{3,3} & [a] & O_{3,3} & [b] & [c] & O_{3,3} & O_{3,3} \\
O_{3,3} & [c] & O_{3,3} & [a] & [b] & [c] & O_{3,3}
\end{bmatrix}, \tag{22}
\]

\[
G = \begin{bmatrix}
0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3}
\end{bmatrix}, \tag{23}
\]

\[
\alpha = bdiag(\ell_{q_d}^{n_{Q}}) \times \gamma, \quad \beta = bdiag(\ell_{q_d}^{n_{Q}}), \tag{24}
\]

\[
\beta = bdiag(\ell_{q_d}^{n_{Q}}), \quad \gamma = bdiag(-\Omega_{i,j}^{n_{Q}} - \Omega_{i,j}^{n_{Q}} - \Omega_{i,j}^{n_{Q}} - \Omega_{i,j}^{n_{Q}} - \Omega_{i,j}^{n_{Q}})
\]

\[
\nu = bdiag(J_i^{-1}(\ell_{q_d}^{n_{Q}}) - \ell_{q_d}^{n_{Q}}), \quad \omega = bdiag(\ell_{q_d}^{n_{Q}})
\]

\[
a_i = M_{(\delta v)}(\omega_{id} - \omega_{id})^2 \beta_{i,j} \tag{25}
\]

\[
b_i = M_{(\delta v)}(\omega_{id} - \omega_{id})^2 \beta_{i,j}
\]

\[
e_{i,j} = \begin{cases}
\frac{M_{(\delta v)}(\omega_{id} - \omega_{id})^2}{\delta_{i,j}} & \text{if } i = j, \\
0 & \text{if } i \neq j,
\end{cases}
\]

\[
d_{ij} = \begin{cases}
0 & \text{if } i = j, \\
M_{i,j} & \text{if } i \neq j
\end{cases}
\]

\[
e_{i,j} = -f_{j_d} R_{j_d} \delta_{i,j}, \quad j \in \mathcal{I}
\]

\[
f_{ij} = \begin{cases}
0 & \text{if } i \neq j, \\
\phi_{i,j} R_{j_d} \delta_{i,j} & \text{if } i = j.
\end{cases}
\]

\[
g_{ij} = R_{j_d} \delta_{i,j}, \quad \phi_{i,j} \in \mathbb{R}^{3}
\]

and \(bdiag\) is block diagonal matrix. Note that \(M_{(\delta v)}\) in (19), (20) is the same mass matrix in (2), except is the function of desired link attitudes \(\{\ell_{i,j}\}\).

As seen, (17)-(18) is a time-varying constrained linear system. The constraints arise due to the variation constraint on \(S^2\) as discussed in Wu and Sreenath (2015). Controllability of the constrained linear equation can be shown similar to Wu and Sreenath (2015), however, due the complexity of the matrices \(A, B, C\) computing the controllability matrix would be intractable.

4.2 Finite-Horizon LQR

Assuming, we have the complete reference trajectory we can implement any linear control technique for (17)-(18). Similar to (Wu and Sreenath, 2015, Lemma 1), we can show that the constraint (18) is time-invariant, i.e., if the initial condition satisfies the constraint, solution to the linear system would satisfy the constraint for all time. However, due to this constraint, the controllability matrix computed using \(A, B\) might not be full-rank and requires state transformation into the unconstrained space to result in full-rank controllability matrix.

Instead, we opt for a finite-horizon LQR controller for the variation-linearized dynamics about a time-varying desired trajectory. We chose a finite-time horizon \(T\), the terminal cost matrix \(P_T\) and pick cost matrices for states \(Q_1 = Q_T^T\) and inputs \(R_1 = R_T^T\). Finally, we solve the continuous-time Riccati equation backwards in time to obtain the gain matrix \(P(t)\), that satisfies,

\[
\dot{P} = -Q_1 -PBQ_2^T \quad \text{and } \quad \dot{u} = -K \delta x.
\]

Since the gains are computed backwards in time, the computed input would result in a stable control for the constrained linear-system. The net control-input to the nonlinear system can be computed as,

\[
u(t) = u(t) + \delta u.
\]

In the next section, we present few numerical simulations with the finite-horizon LQR performing tracking control on the full nonlinear-dynamics.
LQR discussed in the previous section and dynamics of degrees of freedom and under-actuation for this setup are with the cable hanging between these two points. De-
oscillations are not accounted for in the control and can
cable orientation results in oscillations in the cable. These
and similar quadrotor attitude error is defined as,
error in the cable orientation and the resulting error plots
due to the cable at steady state. We start with some initial
controllers on the two quadrotors with feed-forward forces
and the setpoint is given as,
considered for the simulations,
ii. Three Quadrotor system: Setpoint tracking for cable
suspended from three-quadrotors is presented here. Pa-
rameters for the system are as follows,
and #DOF = 32, #DOuA = 20. Various tracking errors
for the system are presented in Figure 5 and snapshots for
the system are shown in Figure 6.

5.2 Trajectory Tracking
In this section, we show that the presented controller
tracks a desired time-varying trajectory with initial errors.
We use the following system parameters,
and the rest same as those given in Section 5.1. We
consider the following flat output trajectory,
Rest of the states and inputs can be computed using
differential-flatness. We use the linearized-dynamics and
the finite-horizon LQR presented in the previous sections
to achieve the tracking control. Following weights are used
for the LQR,
where \( n_{x} = 6 + 6n_{Q} + 6n_{Q} \). Figure 7 shows snapshots
of the system at different instants along the trajectory. As shown
in the Figure, the proposed controller tracks the desired
trajectory (shown in red) when started with an initial error.

6. RESULTS AND DISCUSSION
Having presented numerical results to validate our
controller, we now present some discussion on limitations and future work.

Limitations
Though increasing discretization helps better represent the
 dynamics of an hose system, it also increases the
computation-complexity. To better study the effect of
discretization we ran multiple simulations with different
discretizations for a fixed cable length and mass. We used

Fig. 6. Snapshots of 3 quadrotor-10 link system while tracking a setpoint. Setpoints for the quadrotor position is shown by the red-hexagons.

Fig. 7. Snapshots of the multiple quadrotors carrying a flexible hose system while tracking the desired trajectory (shown in red) and the resulting trajectory when started with an initial error (shown in black).

Fig. 8. Errors for the trajectory tracking control shown in Fig. 7. $\Psi_q, \Psi_R$ are as defined in (25)-(26).

only control on the quadrotor-positions with feed-forward cable tensions. We used MATLAB 2018a with Intel Core i7-6850K CPU @ 3.60GHz $\times$ 12 to run the simulations. Computation times to simulate 10s for different $n$ are shown in Figure 9. As illustrated in the Figure, computation time increases super-linearly with $n$.

Fig. 9. Computation time to simulate 10s for different discretizations $n$.

While differential-flatness can be used to plan trajectories in the flat-output space and compute desired states and inputs, this computation requires computing $q_i$ and its derivatives from tension $T_i$ and its derivatives, i.e. $q_i = \frac{T_i}{\|T_i\|}$. The complexity of this computation increases for higher-derivatives.

In addition, as listed in the Section 2, we don’t consider the mechanical properties of the hose when deriving the dynamics. Thus, the dynamics derived and the subsequent presented control might not completely capture the system fully and might lead to instability in cases when hose properties are important, such as when water flows in the hose.

Future Work

As part of future work, we would like to address some of the limitations listed in the previous sections, such as, (i) number of discretizations, (ii) number of derivatives to computed, and (iii) water flow in the hose. We would like to study the current system along with all the mechanical properties of the cable and develop controller for such systems. In addition, to implement the control we require state estimation of the cable which as shown is modeled as $(S^2)^n$. Towards this end we hypothesize [Kotaru and Sreenath (2019)] can be extended to estimate the cable state. A possible method to improve the computation time would be to use limited cable states like mid-position of the cable etc., to develop a controller.

7. CONCLUSION

In this work we have studied the multiple quadrotors carrying a flexible hose system. We derived the dynamics by modeling the cable as a series of smaller discrete-links with lumped mass. We derived the coordinate-free dynamics using Langrange-d’Alembert’s principle. We also showed that the given system is differentially-flat, as long as the end of the cable is connect to a quadrotor. Variation-based linearized dynamics were derived about a time-varying desired trajectory. We showed tracking control for the system using finite-horizon LQR for the linear dynamics and validated this through numerical simulations with up-to 10 discretizations of the hose.
Finally, we discussed some of the limitations due to the assumptions and directions for future work.

REFERENCES


Sreenath, K. and Kumar, V. (2013). Dynamics, control and planning for cooperative manipulation of payloads suspended by cables from multiple quadrotor robots. $r_n$, 1(r2), r3.


Appendix A. DYNAMICS DERIVATION

In this section, we present the detailed derivation of the equations of motion, (1)-(3), for the given system. Starting with the principle of least action in (8) and substituting for Lagrangian and virtual work from (6),(7) and (9), we have,
\[
\delta \int \left( \sum_{i \in S} \frac{1}{2} \mathbf{m}_i (v_i, v_i) - \mathbf{m}_i g \mathbf{e}_3 \cdot v_i + \sum_{j \in J} \frac{1}{2} \langle \Omega_j, J \Omega_j \rangle \right) \\
+ \int \left( \sum_{j \in J} \langle W_{ij}, J \rangle + \langle W_{ij}, f_j R_j \mathbf{e}_3 \rangle \right) \, dt. \tag{A.1}
\]

Separating and solving the rotational components, we have the following rotational dynamics:
\[
J_j \dot{\Omega}_j = M - \Omega_j \times J \Omega_j, \quad \forall j \in J. \tag{A.2}
\]

Taking variation on rest of the equation results in,
\[
\delta \int \sum_{i \in S} \left[ \mathbf{m}_i (\delta v_i, v_i) + \delta x_i \cdot \left(-\mathbf{m}_i g \mathbf{e}_3 + f_i R_i \mathbf{e}_3 \mathbf{k}_i \right) \right] \, dt = 0. \tag{A.3}
\]

Expanding the summation,
\[
\delta \int \sum_{i \in S} \left[ \left( \mathbf{m}_i (\delta v_i, v_i) + \delta x_i \cdot u_i \right) + \mathbf{m}_i (\delta v_i, v_i) + \delta x_i \cdot u_i \right] \, dt = 0
\]
Replacing the variations \( \delta v_i, \delta x_i \) with their expansions (12), (13), we get,
\[
\delta \int \left( \left( \mathbf{m}_i (\delta v_i, v_i) + \delta x_i \cdot u_i \right) + \mathbf{m}_i (\delta v_i, v_i) + \delta x_i \cdot u_i \right) \, dt = 0
\]
Using the following simplifications in (A.3),
\[
\begin{align*}
\langle -l_k(q^*_k, \xi_k + \hat{q}_k \cdot \xi_k), v \rangle & = l_k(q^*_k, \xi_k + \hat{q}_k \cdot \xi_k) \cdot \xi_k, \\
\langle -l_k(q^*_k, \xi_k), v \rangle & = l_k(q^*_k, \xi_k) \cdot \xi_k,
\end{align*}
\]
and regrouping the respective variations would result in,
\[
\begin{align*}
\delta x_0 \cdot \left( \sum_{k=0}^{n} \mathbf{m}_k v_k + \delta x_0 \cdot \sum_{k=0}^{n} u_k \right) + \\
\xi_1 \cdot \left( l_1 q_1^* \sum_{k=1}^{n} \mathbf{m}_k v_k + \right) \\
+ \xi_1 \cdot \left( l_1 q_1^* \sum_{k=1}^{n} \mathbf{m}_k v_k + \right) \\
\xi_n \cdot \left( \mathbf{m}_n l_n (q_n^* v_n) + \right) \\
\xi_n \cdot \left( \mathbf{m}_n l_n (q_n^* u_n) \right)
\end{align*}
\]
Integration by parts on the respective variation sets results in
\[
\delta \int \left( \left( \mathbf{m}_i (\delta v_i, v_i) + \delta x_i \cdot u_i \right) + \\
- \xi_1 \cdot \left( l_1 q_1^* \sum_{k=1}^{n} \mathbf{m}_k v_k + l_1 q_1^* \sum_{k=1}^{n} \mathbf{m}_k v_k \right) \\
+ \xi_1 \cdot \left( l_1 q_1^* \sum_{k=1}^{n} \mathbf{m}_k v_k + l_1 q_1^* \sum_{k=1}^{n} \mathbf{m}_k v_k \right) \\
- \xi_n \cdot \left( l_n q_n^* \mathbf{m}_n v_n \right) + \xi_n \cdot \left( l_n q_n^* \mathbf{m}_n u_n \right)
\right) \, dt = 0. \tag{A.4}
\]

By principle of least action the above integral is valid \( \forall \delta x_0, \xi_1, t \) and thus, to ensure the above equation to be zero for all time we have,
\[
\begin{align*}
\langle -\left( \mathbf{m}_0 v_0 + \mathbf{m}_1 v_1 + \ldots + \mathbf{m}_n v_n \right) + \sum_{k=0}^{n} \xi_1 & = 0, \\
q_1 & = 0, \\
l_1(q_1^*)^2 & = \mathbf{m}_0 (v_0 + \sum_{k=1}^{n} l_k q_k) = \sum_{k=1}^{n} \xi_1
\end{align*}
\]
\[
\begin{align*}
q_0 & = 0, \\
l_0(q_0^*)^2 & = \mathbf{m}_0 (v_0 + \sum_{k=1}^{n} l_k q_k)
\end{align*}
\]
Simplifying the above equations using (14) and the following relations,
\[
\begin{align*}
\dot{q} & = \omega \times q, \\
\ddot{q} & = \dddot{q} + \omega \times \dddot{q} = \omega \times q - \|\omega\|^2 q
\end{align*}
\]
\[
(q^*)^2 \ddot{q} = q \times (q \times q) = (q \cdot \dddot{q}) q - (q \cdot q) \dddot{q} = - (\dddot{q} \cdot \dddot{q}) q - \dddot{q},
\]
would result in (2).

### Appendix B. DIFFERENTIAL FLATNESS

In this section, we present the proof for differential-flatness stated in Lemma 1

**Proof.** Illustration of the differential-flatness is shown in Figure 2. For the purpose of proving differential-flatness we redefine the dynamics of the system using tensions in the cable links as given below,
\[
\begin{align*}
\mathbf{m}_0 \ddot{x}_0 &= T_1 - \mathbf{m}_0 g \mathbf{e}_3, \\
\mathbf{m}_a \ddot{x}_a &= T_{a+1} - T_a - \mathbf{m}_a g \mathbf{e}_3, \\
\mathbf{m}_b \ddot{x}_b &= T_{b+1} - T_b - \mathbf{m}_a g \mathbf{e}_3 + f_j R_j \mathbf{e}_3, \\
\mathbf{m}_b \ddot{x}_b &= -T_0 - \mathbf{m}_a g \mathbf{e}_3 + f_j R_j \mathbf{e}_3,
\end{align*}
\]
where \( \forall a \in S \setminus \{0, I\} \) i.e., all the points excluding the starting point of the cable and those connected to the quadrotors and \( \forall b \in I \setminus \{n\} \) and the quadrotor attitude dynamics are as given in (3). Also note \( n \in I, \) i.e., end the cable is attached to the quadrotor. Number of inputs in the system are \( 4nQ + 3 \) (for position \( x_0 \)) + \( 3nQ - 1 \) (3 for each tension \( T_{k+1} \forall k \in I \setminus \{n\} \) and \( nQ \) for each quadrotor yaw \( \psi_j \)) = \( 4nQ \).

Making use of the these dynamics we prove the flatness as follows.

(i) Given, \( x_0 \) is a flat-output and therefore we have the cable start position and its derivatives as shown,
\[
\begin{align*}
\{ x_0, \dot{x}_0, \ddot{x}_0, x_0(3), \ldots, x_0(2n+4) \}. \tag{B.5}
\end{align*}
\]
(ii) Taking derivatives of (B.1) and making use of (B.5) we have tension vector \(T_1\) in the first link and its derivatives,
\[
\{ T_1, \dot{T}_1, \ddot{T}_1, T_1^{(1)}, \ldots, T_1^{(2n+2)} \}. \tag{B.6}
\]
(iii) Attitude of the first link is then determined from the tension vector in (B.6) as
\[
q_1 = \frac{T_1}{||T_1||} \tag{B.7}
\]
and its higher derivatives,
\[
\{q_1, \ldots, q_1^{(2n+2)}\}, \tag{B.8}
\]
are computed by taking derivatives of (B.7) and using (B.6).
(iv) Position and its derivatives of the next link point-
\[
mass \overline{m}_1 \text{ is computed using (4),}
\]
\[
\{x_1, \dot{x}_1, \ddot{x}_1, \ldots, x_1^{(2n+2)}\}. \tag{B.9}
\]
(v) Repeating the steps (ii)-(iv) we can compute the link
attitudes, tensions and the positions iteratively till \(x_n\).
(vi) Using (B.3) and the fact that \(T_{b+1}\) is a flat-output
(note \(b \in T_1\{n\}\)) we can compute the thrust in the
quadrotor \(f_bR_t\).\(\vec{e}_2\)
(vii) From \(x_b\), \(f_bR_t\) and their derivatives, the quadrotor
attitude, angular velocity \(R_t\), \(\Omega_j\) and moment \(M_j\)
can be computed as shown in [Mellinger and Kumar
(2011)].
(viii) Rest of the states and inputs for the multiple quadro-
tors carrying a flexible hose segments can be iteratively
determined as described above.

Proof for Corollary 2 is given below,

**Proof.** For tethered system we have \(x_0 \equiv 0\) and \(T_1\) is
known since it is a flat-output, i.e., steps (i)-(ii) (see (B.5)-(B.6)). Rest of the proof follows form Lemma 1.

Appendix C. VARIATION-BASED LINEARIZED DYNAMICS

Taking variations with respect to desired states for various
states is as follows,
\[
\delta q_i = \xi_i q_i = -q_i \delta \xi_i \tag{C.1}
\]
\[
\delta (||\omega_i||^2) = \delta (\omega_i^\mathsf{T} \omega_i) = 2\omega_i \delta (\omega_i) \tag{C.2}
\]
\[
\delta R_j = R_j \delta \Omega_j \tag{C.3}
\]
Taking variation on the first row of (2),
\[
\left( M_{dd} \lambda_3 \dot{x}_0 - \sum_{i=1}^n M_{0i} q_i^\mathsf{T} \dot{\omega}_i - \sum_{i=1}^n M_{0i} ||\omega_i||^2 q_i + \sum_{k=0}^{n} u_k \right) = 0 \tag{C.4}
\]
\[
M_{dd} \lambda_3 \delta \dot{x}_0 - \sum_{i=1}^n M_{0i} q_i^\mathsf{T} \delta \dot{\omega}_i =
\sum_{i=1}^n \left( M_{0i} \left[ (\omega_{id}^\mathsf{T} - ||\omega_{id}||^2 I) q_i^\mathsf{T} \right] \xi_i + M_{0i} (2\omega_{id} \omega_{id}^\mathsf{T}) \delta \omega_i \right)
+ \sum_{k=0}^{n} \left( (\delta f_k) R_{kd} e_k^\mathsf{T} \right) 1_k + \sum_{k=0}^{n} M_{kk} (2\omega_{kd} \omega_{kd}^\mathsf{T}) \delta \omega_k \right) \tag{C.5}
\]
\[
\delta \left( M_{00} I_3 \dot{\omega}_0 - \sum_{i=1}^n M_{0i} q_i^\mathsf{T} \omega_i = \sum_{i=1}^n M_{0i} ||\omega_i||^2 q_i + \sum_{k=0}^{n} u_k \right) \tag{C.6}
\]
\[
M_{00} I_3 \delta \dot{\omega}_0 - \sum_{i=1}^n M_{0i} q_i^\mathsf{T} \delta \omega_i =
\sum_{i=1}^n \left( M_{0i} \left[ (\omega_{id}^\mathsf{T} - ||\omega_{id}||^2 I) q_i^\mathsf{T} \right] \xi_i + M_{0i} (2\omega_{id} \omega_{id}^\mathsf{T}) \delta \omega_i \right)
+ \sum_{k=0}^{n} \left( (\delta f_k) R_{kd} e_k^\mathsf{T} \right) 1_k + \sum_{k=0}^{n} M_{kk} (2\omega_{kd} \omega_{kd}^\mathsf{T}) \delta \omega_k \right) \tag{C.7}
\]
\[
\delta M_{0i} \delta q_i + M_{ii} \delta \dot{\omega}_0 - \sum_{j=1, j \neq i}^n M_{ij} q_j^\mathsf{T} \delta \dot{\omega}_j =
\sum_{k=1}^n \left( M_{0k} ||\omega_k||^2 q_k + l_i (q_i^\mathsf{T}) \sum_{k=1}^n u_k \right) \tag{C.8}
\]
\[
\delta M_{0k} \delta q_k + M_{kk} \delta \dot{\omega}_0 - \sum_{j=1, j \neq k}^n M_{kj} q_j^\mathsf{T} \delta \dot{\omega}_j =
\sum_{k=1}^n \left( M_{0k} ||\omega_k||^2 q_k + l_i (q_i^\mathsf{T}) \sum_{k=1}^n u_k \right) \tag{C.9}
\]
\[
\delta M_{00} I_3 \delta \omega_0 - \sum_{i=1}^n M_{0i} q_i^\mathsf{T} \delta \omega_i =
\sum_{i=1}^n \left( M_{0i} \left[ (\omega_{id}^\mathsf{T} - ||\omega_{id}||^2 I) q_i^\mathsf{T} \right] \xi_i + M_{0i} (2\omega_{id} \omega_{id}^\mathsf{T}) \delta \omega_i \right)
+ \sum_{k=0}^{n} \left( (\delta f_k) R_{kd} e_k^\mathsf{T} \right) 1_k + \sum_{k=0}^{n} M_{kk} (2\omega_{kd} \omega_{kd}^\mathsf{T}) \delta \omega_k \right) \tag{C.10}
\]
From (C.5) & (C.10), we have,
\[
\begin{bmatrix}
M_{00}I_3 & -q_{1d}^x M_{01} & -q_{2d}^x M_{02} & \ldots & -q_{n_d}^x M_{0n} \\
M_{10q_1^x} & M_{11}I_3 & -M_{12} q_{1d}^x q_{2d}^x & \ldots & -M_{1n} q_{1d}^x q_{n_d}^x \\
M_{20q_2^x} & -M_{21} q_{2d}^x q_{1d}^x & M_{22}I_3 & \ldots & -M_{2n} q_{2d}^x q_{n_d}^x \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
M_{n0q_n^x} & -M_{n1} q_{n_d}^x q_{1d}^x & -M_{n2} q_{n_d}^x q_{2d}^x & \ldots & M_{nn}I_3
\end{bmatrix}
\]

\[
\begin{bmatrix}
[O \ a_1 \ldots \ a_n \ O \ b_1 \ldots \ b_n \ e_1 \ldots \ e_{n_Q} \ O \ldots \ O] \\
[O \ c_{11} \ldots \ c_{1n} \ O \ d_{11} \ldots \ d_{1n} \ f_1 \ldots \ f_{1n} \ O \ldots \ O] \\
[\vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots] \\
[O \ c_{n1} \ldots \ c_{nn} \ O \ d_{n1} \ldots \ d_{nn} \ f_{n_Q} \ldots \ f_{n_Qn} \ O \ldots \ O]
\end{bmatrix}
\]

\[\delta \dot{\mathbf{v}}, \quad (C.11)\]

where

\[a_i = M_{0i} \left( \dot{\omega}_d^x + \| \omega_d \|^2 I_3 \right) q_i^x \]

\[b_i = M_{0i} (2q_{id}^x \omega_d^x), \quad i = \{1, \ldots, n\} \]

\[c_{ij} = \begin{cases} 
M_{i0} \dot{\omega}_d^x - \sum_{j=1, j \neq i}^{n} M_{ij} (q_{jd}^x \omega_d^x) + \| \omega_d \|^2 q_{jd}^x - l_i \left( \sum_{k=1}^{n} u_k^x \right) (-q_i^x) & \text{if } j = i \\
M_{ij} q_{id}^x (\dot{\omega}_d^x - \| \omega_d \|^2 I_3) q_{jd}^x & \text{if } j \neq i 
\end{cases} \]

\[d_{ij} = \begin{cases} 
O & \text{if } j = i \\
M_{ij} [2q_{id}^x q_{jd}^x (\omega_d^x)] & \text{if } j \neq i 
\end{cases} \]

\[e_j = -f_{jd} R_{jd} e_3^x, \quad j \in \mathcal{I} \]

\[f_{ij} = \begin{cases} 
[0] & \text{if } j \notin \mathcal{I} \\
(-l_i q_i^x) f_{jd} R_{jd} e_3^x, & \text{if } j \in \mathcal{I}, j \geq i \\
O_{3,3} & \text{if } j \in \mathcal{I}, j < i 
\end{cases} \]

\[g_j = R_{jd} e_3 \]

\[h_{ij} = \begin{cases} 
[l_i q_i^x] R_{jd} e_3, & \text{if } j \notin \mathcal{I} \\
O_{3,1} & \text{if } j \in \mathcal{I}, j \geq i \\
O_{3,1} & \text{if } j \in \mathcal{I}, j < i 
\end{cases} \]
\[ \delta x = [\delta x, \, \xi_1, \, \ldots, \, \xi_n, \, \delta v, \, \delta \omega_1, \, \ldots, \, \delta \omega_n, \, \eta_{j1}, \, \ldots, \, \eta_{jnQ}, \, \delta \Omega_{j1}, \, \ldots, \, \delta \Omega_{jnQ}]^T \]

\[ \delta u = [f_{j1}, \, f_{j2}, \ldots, \, f_{jnQ}, \, M_{j1}^T, \, M_{j2}^T, \ldots, \, M_{jnQ}]^T \]

\[ \delta x = A\delta x + B\delta u \]

where,

\[
A = \begin{bmatrix}
0_{3,3} & 0_{3,3n} & I_{3,3} & 0_{3,3n} & 0_{3,3nQ} & 0_{3,3nQ} \\
0_{3n,3} & \alpha & 0_{3,3} & 0_{3,3n} & 0_{3,3nQ} & 0_{3,3nQ} \\
0_{3nQ,3} & 0_{3nQ,3n} & 0_{3nQ,3} & 0_{3nQ,3n} & \gamma & 0_{3nQ,3nQ} \\
0_{3nQ,3} & 0_{3nQ,3n} & 0_{3nQ,3} & 0_{3nQ,3n} & 0_{3nQ,3nQ} & \nu
\end{bmatrix}, \quad B = \begin{bmatrix}
O_{3(n+1),4nQ} \\
O_{3nQ,4nQ} \\
O_{3nQ,4nQ} \\
O_{3nQ,4nQ}
\end{bmatrix}
\]

\[\mathbf{M} = \begin{bmatrix}
M_{00}I_3 & -q_{1d}^x M_{01} & \cdots & -q_{n_{nd}}^x M_{0n} \\
-M_{00}q_{1d}^y & -M_{11}I_3 & \cdots & M_{1n}q_{n_{nd}} \\
\vdots & \vdots & \ddots & \vdots \\
-M_{n0}q_{1d}^y & M_{n1}q_{n_{nd}} & \cdots & -M_{nn}I_3
\end{bmatrix}
\]

\[\mathbf{F} = \begin{bmatrix}
O_{3,3} [a_i] & O_{3,3} [b_i] & O_{3,3} [c_j] & O_{3mQ,3nQ} \\
O_{3n,3} [d_{i,j}] & O_{3n,3} [e_{i,j}] & f_r & O_{3nQ,3nQ}
\end{bmatrix}
\]

\[\mathbf{G} = \begin{bmatrix}
[\breve{\gamma}_i] & O_{3(n+1),3nQ}
\end{bmatrix}
\]