L_1 Adaptive Control Barrier Functions for Nonlinear Underactuated Systems

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***Abstract*— This paper presents a novel integration of adaptive control and control barrier functions that offers tracking stability as well as safety-critical constraints for nonlinear underactuated systems in the presence of model uncertainty. The proposed method is based on L_1 adaptive control with a nonlinear closed-loop reference model based on control barrier functions. For underactuated systems, adaptation based on the control output does not imply adaptation in the system state. Therefore, to guarantee adaptive constraint enforcement which depends on the entire state, we introduced a modified reference state to represent the zero dynamics or internal states of the real system inside the reference model. We evaluate our proposed control design for the problem of dynamic walking of an underactuated bipedal robot subject to safety-critical constraints of foot placements on stepping stones under significant model uncertainty. We present numerical results on RABBIT, a five-link planar bipedal robot carrying a large unknown load on its torso. Our proposed controller is able to demonstrate walking while strictly enforcing the above constraints with an unknown load of up to 30 Kg (94% of the robot mass).**

I. INTRODUCTION

Stability and safety are two major criteria of a control system for robotics applications. Recently, the use of the quadratic program (QP)-based control via control Lyapunov functions (CLFs) [10] and control barrier functions (CBFs) [2] has become increasingly popular [3]. This framework enables handling safety-critical constraints effectively in real-time. Experimental validation of this type of controller for the problem of Adaptive Cruise Control was presented in [15]. This framework has also been extended to various interesting application domains, such as safety-critical geometric control for quadrotor systems [24] and safety-critical dynamic walking for bipedal robots [20], [11]. Although this work can handle safety-critical constraints, however a precise model of the system is required to enforce the constraints.

Moreover, as presented in [25], preliminary robustness analysis of the CBFs indicates that the safety-critical constraint will be violated in the presence of model uncertainty, with the amount of violation being bounded by the value of the upper bound of the model uncertainty. In particular, model uncertainty leads to constraint violation of the safety-critical constraints. Recently, there are several efforts in developing adaptive control barrier functions for parametric uncertainty [23] or using data-driven approaches [14], [9]. In addition, these works also do not apply to under-actuated systems.

L_1 adaptive control technique has enabled decoupling of adaptation and robustness in adaptive control techniques, guaranteeing not only stability [5] but also transient performance [6]. L_1 adaptive control appears to have great potential for application in aerospace systems, illustrated in [7], [22]. The presence of a low-pass filter in the L_1 adaptive control allows us to prevent high-frequency control signals that are typical and frequently seen in adaptive control problems. This will be critical to keeping motor torques less noisy and will contribute to ensuring the validity of the unilateral ground contact constraints, as well as retaining the energy efficiency of walking control of bipedal robots. In our prior work on L1 adaptive CLF [18], we have presented an adaptive control framework for model uncertainty with a nonlinear reference model that arises as a closed-loop system controlled by a control Lyapunov function based quadratic program [1], [10]. In this paper, we build off this prior work to simultaneously handle adaptive stability and state-dependent constraints in the presence of model uncertainty for both fully-actuated and under-actuated systems. We will do this by incorporating L_1 adaptive control with the CLF and CBF control framework. In addition, for underactuated systems, because the dimension of the control inputs and outputs is smaller than that of the system state, there always exists internal state or zero dynamics that we don't directly control in the system (e.g., the phase variable in control of bipedal robots using HZD [1]). As a result, adaptive control based on control output does not guarantee adaptation for the entire system state for an underactuated system. Therefore, we also propose to incorporate internal states from the real system into the reference model to guarantee adaptation for the safety constraints that depend on the entire system state.

The main contributions of this paper with respect to prior work are as follows:

- Introduction of a new adaptive control technique that handles stability and safety constraints under high levels of model uncertainty.
- Adaptation for under-actuated systems by incorporating internal states from the real system into the reference model.
- Numerical validation of the proposed controllers on:
  - Control of a rectilinear spring-cart system.
  - Dynamic walking of a bipedal robot while carrying an unknown load, subject to contact force constraints and precise foot-step location constraints.

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- Our proposed controller enables a simulation model of RABBIT to dynamically walk on stepping stones while carrying an unknown load of up to 30 Kg (94% of the robot mass).
- We have shown significant improvement in the adaptability of the approach in comparison with the non-adaptive baseline framework and our prior work on robust CBF [17].

The rest of the paper is organized as follows. Section II revisits control barrier functions and control Lyapunov functions based quadratic programs (CBF-CLF-QPs). Section III discusses the adverse effects of uncertainty and then Section IV presents the proposed L_1 adaptive control framework for CBF-CLF-QP. Section V presents numerical validation on different dynamical robotic systems. Finally, Section VI provides concluding remarks.

II. Control Lyapunov Functions and Control Barrier Function based Quadratic Programs Revisited

A. System Model and Input-Output Linearizing Control

Consider a nonlinear control affine model

$$\begin{cases} \dot{x} &= f(x) + g(x)u, \\ y &= y(x), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input, and $y \in \mathbb{R}^m$ is a set of outputs.

If the control output $y(x)$ has relative degree 2, then the time-derivative $\dot{y}(x)$ will be a function of the state x and not dependent on the control input u. Considering the second time-derivative $\ddot{y}$, we have:

$$\ddot{y} = \frac{\partial \dot{y}}{\partial x}\dot{x} = L_f^2 y(x) + L_g L_f y(x) u. \tag{2}$$

where L represents the Lie derivative. To be more specific:

$$L_f^2 y(x) \triangleq \frac{\partial \dot{y}}{\partial x} f(x), \qquad L_g L_f y(x) \triangleq \frac{\partial \dot{y}}{\partial x} g(x). \tag{3}$$

If the decoupling matrix $L_g L_f y(x)$ is invertible, then the controller

$$u(x,\mu) = u_{ff}(x) + (L_g L_f y(x))^{-1}\mu, \tag{4}$$

with the feed-forward control input

$$u_{ff}(x) = -(L_g L_f y(x))^{-1} L_f^2 y(x), \tag{5}$$

input-output linearizes the system. The dynamics of the system (1) can then be described in terms of dynamics of the transverse variables, $\eta \in \mathbb{R}^{2m}$, and the coordinates $\xi \in \mathcal{Z}$ with $\mathcal{Z}$ being the co-dimension $2m$ manifold

$$\mathcal{Z} = \{x \in \mathbb{R}^n \mid \eta(x) \equiv 0\}. \tag{6}$$

One choice for the transverse variables is,

$$\eta = \begin{bmatrix} y(x) \\ \dot{y}(x) \end{bmatrix}. \tag{7}$$

The input-output linearized system then is,

$$\begin{cases} \dot{\eta} &= \bar{f}(\eta) + \bar{g}(\eta)\mu \\ \dot{\xi} &= p(\eta,\xi) \\ y &= y(\eta), \end{cases} \tag{8}$$

where ξ represents uncontrolled states [1], and

$$\bar{f}(\eta) = F\eta, \quad \bar{g}(\eta) = G, \tag{9}$$

with,

$$F = \begin{bmatrix} O & I \\ O & O \end{bmatrix} \text{ and } G = \begin{bmatrix} O \\ I \end{bmatrix}. \tag{10}$$

The linear system in (9) is in controllable canonical form, and a linear controller such as $\mu = -K\eta$ can be designed such that the closed-loop system $\dot{\eta} = (F - GK)\eta$ is stable. A corresponding quadratic Lyapunov function can then be established through the Lyapunov equation.

B. Control Lyapunov Function based Quadratic Programs

1) CLF-QP: Instead of a linear control design $\mu = -K\eta$ in (4), an alternative control design is through a control Lypapunov function $V(\eta)$, wherein a control is chosen point-wise in time such that the time deriviative of the Lyapunov function $\dot{V}(\eta,\mu) \leq 0$, resulting in stability in the sense of Lyapunov, or $\dot{V}(\eta,\mu) < 0$ for asymptotic stability, or $\dot{V}(\eta,\mu) + \lambda V(\eta) \leq 0, \lambda > 0$ for exponential stability.

To enable directly controlling the rate of convergence, we use a *rapidly exponentially stabilizing control Lyapunov function (RES-CLF)*, introduced in [1]. RES-CLFs provide guarantees of *rapid exponential stability* for the transverse variables η. In particular, a function $V_\varepsilon(\eta)$ is a RES-CLF for the system (1) if there exist positive constants $c_1, c_2, c_3 > 0$ such that for all $0 < \varepsilon < 1$ and all states (η, z) it holds that

$$c_1\|\eta\|^2 \leq V_\varepsilon(\eta) \leq \frac{c_2}{\varepsilon^2}\|\eta\|^2, \tag{11}$$

$$\dot{V}_\varepsilon(\eta,\mu) + \frac{c_3}{\varepsilon}V_\varepsilon(\eta) \leq 0. \tag{12}$$

The RES-CLF will take the form:

$$V_\varepsilon(\eta) = \eta^T \begin{bmatrix} \frac{1}{\varepsilon}I & 0 \\ 0 & I \end{bmatrix} P \begin{bmatrix} \frac{1}{\varepsilon}I & 0 \\ 0 & I \end{bmatrix} \eta =: \eta^T P_\varepsilon \eta, \tag{13}$$

and the time derivative of the RES-CLF (13) is computed as

$$\dot{V}_\varepsilon(\eta,\mu) = \frac{\partial V_\varepsilon}{\partial \eta}\dot{\eta} = L_{\bar{f}}V_\varepsilon(\eta) + L_{\bar{g}}V_\varepsilon(\eta)\mu, \tag{14}$$

where

$$\begin{aligned} L_{\bar{f}}V_\varepsilon(\eta) &= \frac{\partial V_\varepsilon}{\partial \eta}\bar{f}(\eta) = \eta^T(F^T P_\varepsilon + P_\varepsilon F)\eta, \\ L_{\bar{g}}V_\varepsilon(\eta) &= \frac{\partial V_\varepsilon}{\partial \eta}\bar{g}(\eta) = 2\eta^T P_\varepsilon G. \end{aligned} \tag{15}$$

It can be show that for any Lipschitz continuous feedback control law μ that satisfies the RES condition (12), it holds that

$$V(\eta) \leq e^{-\frac{c_3}{\varepsilon}t}V(\eta(0)), \quad \|\eta(t)\| \leq \frac{1}{\varepsilon}\sqrt{\frac{c_2}{c_1}}e^{-\frac{c_3}{2\varepsilon}t}\|\eta(0)\|,$$

i.e. the rate of exponential convergence can be directly controlled with the constant ε through $\frac{c_3}{\varepsilon}$. One such controller is the CLF-based quadratic program (CLF-QP)-based controller, introduced in [10], where μ is directly selected through an online quadratic program to satisfy (12), with additional input constraints such as input saturation, friction constraints, contact force constraints, etc., for robotic locomotion and manipulation.

The CLF-QP based controller with additional constraints then takes the form,

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CLF-QP with Constraints:

$$
\begin{aligned}
u^*(x) = &\underset{u,\mu,\delta}{\operatorname{argmin}} && \mu^T\mu + p\delta^2 && (16)\\
&\text{s.t.} && \dot{V}_\varepsilon(\eta,\mu) + \frac{c_3}{\varepsilon}V_\varepsilon(\eta) \le \delta, && \textbf{(CLF)}\\
& && A_c(x)u \le b_c(x), && \textbf{(Constraints)}\\
& && u = u_{ff}(x) + (L_gL_fy(x))^{-1}\mu, && \textbf{(IO)}
\end{aligned}
$$

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where p is a large positive number that represents the penalty of relaxing the inequality, and $A_c(x), b_c(x)$ are formulated based on addition input constraints.

Having revisited control Lyapunov function based quadratic programs, we will next revisit control Barrier functions.

C. Control Barrier Function

We begin with the control affine system (1) with the goal to design a controller to keep the state x in the safe set

$$\mathcal{C} = \{x \in \mathbb{R}^n : h(x) \ge 0\}, \tag{17}$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function. Then a function $B : \mathcal{C} \to \mathbb{R}$ is a Control Barrier Function (CBF) [2] if there exists class $\mathcal{K}$ function α_1 and α_2 such that, for all $x \in Int(\mathcal{C}) = \{x \in \mathbb{R}^n : h(x) > 0\}$,

$$\frac{1}{\alpha_1(h(x))} \le B(x) \le \frac{1}{\alpha_2(h(x))}, \tag{18}$$

$$\dot{B}(x,u) \le \frac{\gamma}{B(x)}, \tag{19}$$

where

$$\dot{B}(x,u) = \frac{\partial B}{\partial x}\dot{x} = L_fB(x) + L_gB(x)u, \tag{20}$$

with the Lie derivatives computed as,

$$L_fB(x) = \frac{\partial B}{\partial x}f(x), \qquad L_gB(x) = \frac{\partial B}{\partial x}g(x). \tag{21}$$

Thus, if there exists a Barrier function $B(x)$ that satisfies the CBF condition in (19), then $\mathcal{C}$ is forward invariant, or in other words, if $x(0) = x_0 \in \mathcal{C}$, i.e., $h(x_0) \ge 0$, then $x = x(t) \in \mathcal{C}, \forall t$, i.e., $h(x(t)) \ge 0, \forall t$. Note that, as mentioned in [2], this notion of a CBF is stricter than standard notions of CBFs in prior literature that only require $\dot{B} \le 0$.

In this paper, we will use the following reciprocal control Barrier candidate function:

$$B(x) = \frac{1}{h(x)}. \tag{22}$$

Incorporating the CBF condition (19) into the CLF-QP, we have the following CBF-CLF-QP based controllers:

CBF-CLF-QP:

$$
\begin{aligned}
u^*(x) = &\underset{u,\mu,\delta}{\operatorname{argmin}} && \mu^T\mu + p\ \delta^2 && (23)\\
&\text{s.t.} && \dot{V}_\varepsilon(\eta,\mu) + \frac{c_3}{\varepsilon}V_\varepsilon(\eta) \le \delta, && \textbf{(CLF)}\\
& && \dot{B}(x,u) - \frac{\gamma}{B(x)} \le 0, && \textbf{(CBF)}\\
& && A_c(x)u \le b_c(x), && \textbf{(Constraints)}\\
& && u = u_{ff}(x) + (L_gL_fy(x))^{-1}\mu. && \textbf{(IO)}
\end{aligned}
$$

As presented in [2], the standard CBF is for the velocity based safety constraints, or constraints such as (17) with $h(x)$ that has relative-degree one. For applications with constraints of relative degree two or higher, we can use extension in [24] or Exponential Control Barrier Functions [21].

Having presented control Lyapunov functions, control Barrier functions, and their incorporation into a quadratic program with constraints, we now explore the effects of model uncertainty.

III. Adverse Effects of Uncertainty in Dynamics on the CBF-CLF-QP controller

The CBF-CLF-QP controller presented in Section II is a powerful method that has been deployed successfully for different applications, for example Adaptive Cruise Control [15], quadrotor flight [24], and dynamic walking for bipedal robots [20], [11].

However, this nonlinear control approach requires the knowledge of an accurate dynamical model to guarantee stability and safety for the system. To be more specific, uncertainty in the nonlinear functions $f(x), g(x)$ of the dynamics will affect the calculation of the Lie derivatives in (3), (15) and (21). Therefore, the controller no longer guarantees the CLF and CBF conditions in the QP (23). Therefore, model uncertainty that is usually present in physical systems can potentially cause poor quality of control leading to tracking errors, and potentially leading to instability [19], as well as violation of the safety-critical constraints [25]. In this section, we will explore the effect of uncertainty on the CLF-QP controller, and safety constraints enforced by the CBF-QP controller.

A. Effects of uncertainty on CLFs

In order to analyze the effect of model uncertainty on our controllers, we assume that the vector fields, $f(x), g(x)$ of the real dynamics (1), are unknown. We therefore have to design our controller based on the nominal or reference vector fields $\hat{f}(x), \hat{g}(x)$. A list of different models used in this paper is presented in Table I. Then, the pre-control law (4) get's reformulated as

$$u(x) = \hat{u}_{ff}(x) + (L_{\hat{g}}L_{\hat{f}}y(x))^{-1}\mu, \tag{24}$$

Notations	Model types
f, g	true nonlinear model
$\hat{f}, \hat{g}$	nominal nonlinear model
$\bar{f}, \bar{g}$	true I-O linearized model
$\hat{\bar{f}}, \hat{\bar{g}}$	nominal I-O linearized model

TABLE I: A list of notations for different models used in this paper. A true model represents the actual (possibly not perfectly known) model of the physical system, while the nominal model represents the model that the controller uses. While most controllers assume the true model is known, the robust controllers in this paper use the nominal models and offer robustness guarantees to the uncertainty between the two models.

with

$$\hat{u}_{ff}(x) := -(L_{\hat{g}} L_{\hat{f}} y(x))^{-1} L_{\hat{f}}^2 y(x), \tag{25}$$

where we have used the nominal model rather than the unknown real dynamics.

Substituting $u(x)$ from (24) into (2), the input-output linearized system then becomes

$$\ddot{y} = \mu + \theta, \tag{26}$$

where

$$\begin{aligned} \theta &= \Delta_1 + \Delta_2 \mu, \\ \Delta_1 &= L_f^2 h(x) - L_g L_f h(x) (L_{\hat{g}} L_{\hat{f}} h(x))^{-1} L_{\hat{f}}^2 h(x), \\ \Delta_2 &= L_g L_f h(x) (L_{\hat{g}} L_{\hat{f}} h(x))^{-1} - I. \end{aligned} \tag{27}$$

Remark 1: In the definitions of Δ_1, Δ_2, note that when there is no model uncertainty, i.e., $\hat{f} = f, \hat{g} = g$, then $\Delta_1 = \Delta_2 = 0$.

Using F and G as in (10), the closed-loop system now takes the form

$$\dot{\eta} = F\eta + G(\mu + \theta). \tag{28}$$

B. *Effects of uncertainty on CBFs*

Similar to what we have seen about the effect of uncertainty on CLFs and constraints, we will now see the effect of uncertainty on CBFs. We note that the time-derivative of the Barrier function in (22) depends on the real model. Therefore we need to enforce the following constraint given by (19):

$$\dot{B}(x, f, g, u) = \frac{\partial B}{\partial x}(f(x) + g(x)u) \le \frac{\gamma}{B(x)} \tag{29}$$

where $\dot{x} = f(x) + g(x)u$ is the true system dynamics. As seen in the case of control Lyapunov functions and constraints, naively enforcing this barrier constraint using the nominal model results in,

$$\frac{\partial B}{\partial x}(\hat{f}(x) + \hat{g}(x)u) \le \frac{\gamma}{B(x)} \tag{30}$$

where $\dot{x} = \hat{f}(x) + \hat{g}(x)u$ is the nominal system dynamics known by the controller. Clearly due to model uncertainty, or the significant difference between $(f(x), g(x))$ and

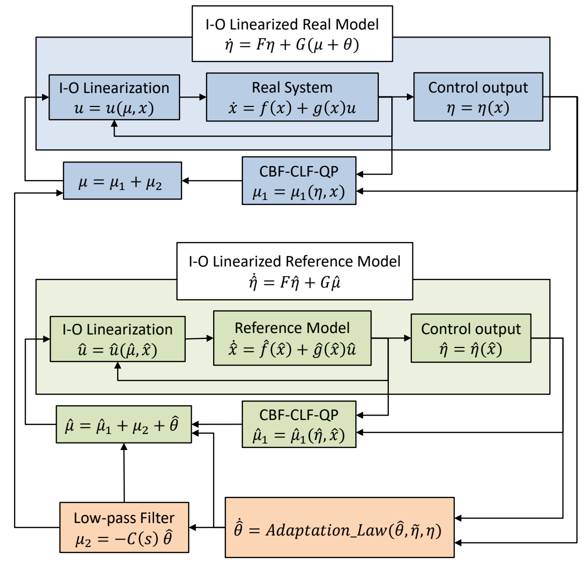


Fig. 1: Control diagram illustrating L_1 adaptive control with a CBF-CLF-QP based closed-loop reference model for fully-actuated systems.

$(\hat{f}(x), \hat{g}(x))$, this constraint is different from the previous one. In fact, as analyzed in [25], this results in violation of the safety-critical constraint established by the Barrier function.

IV. L_1 Adaptive Control Barrier Functions

This Section will present a novel approach called L_1 adaptive control Barrier functions ($L_1 - CBF$) to guarantee safety-critical constraints of nonlinear systems under model uncertainty. Inspired by our work on using L_1 adaptive control with CLF-QP based closed-loop reference model presented in [18] that can be used to guarantee stability under model uncertainty, we now develop the method here to be able to address both stability and constraints under model uncertainty. Our approach works for both actuated and under-actuated systems. Using a reference model with a closed-loop control system controlled by control Barrier function and control Lyapunov function based quadratic program (CBF-CLF-QP), our adaptive control framework drives the performance of the real system including safety and stability close to those of the reference model.

In Section IV-A and IV-B, we will explain our control structure. More details about the control design and safety analysis will be presented later in Section IV-C.

A. *L_1 Adaptive Control Barrier Functions for Fully Actuated Systems*

In this Section, we will propose a new method to enforce both stability and safety-critical constraints under model uncertainty. Our method offers the flexibility to design a closed-loop reference model based on our desired control goal. We therefore can design the reference model so that the desired system state will respect the additional properties (safety-critical constraints in this case) rather than just stability or convergence to the desired trajectory. Note that for the conventional L_1 adaptive control which uses a nominal linear reference model [5], it is not trivial to incorporate additional control goals into the system.

The control diagram of the proposed method on L_1 adaptive control using $CBF-CLF-QP$ based closed-loop reference model is illustrated in Fig. 1. In this framework, the adaptation law will compare the difference between the real system output η (7) and the reference system output $\hat{\eta}$ in order to create an estimation $\hat{\theta}$ of the model uncertainty θ (27). The adaptation law will play the role as a feedback controller that applies $\hat{\theta}$ to drive $\tilde{\eta} = \hat{\eta} - \eta \rightarrow 0$. With fully-actuated systems, driving the control output $\eta \rightarrow \hat{\eta}$ is equivalent to driving the system state $x \rightarrow \hat{x}$. Furthermore, in our proposed method, we also control the reference model to enforce desired constraints or guarantee the reference system state to stay within the safety set. Therefore, as the adaptation law helps to drive the difference between the real state and the reference state $\tilde{x} = \hat{x} - x \rightarrow 0$, the real system state x will follow the same behavior of the reference system sate $\hat{x}$. As a result, the safety-critical constraints of the real system will be guaranteed under model uncertainty. We will formally establish these properties in Sec. IV-C.

B. L_1 Adaptive Control Barrier Functions for Under-Actuated Systems

Since we are interested in applying our controller to a bipedal robot, an under-actuated system, we also want to extend the adaptive CBF framework for under-actuated systems. In this Section, we will introduce a modification of the approach presented in Section IV-A. For under-actuated systems, the dimension of the control output η (7) is less than that of the system state x. The sate can now be written as $x = [\eta; \xi]$, where ξ is the internal state of the under-actuated system. Therefore, while the adaptive control can drive η close to the reference state $\hat{\eta}$, the under-actuated coordinate of the real system may differ from that of the reference model. In other words, the major challenge of this problem is the presence of the internal state ξ that is not a part of the control output η. The adaptive control can help to drive the real control output η to the reference control ouput $\hat{\eta}$. However, because the internal state ξ and $\hat{\xi}$ are not controlled or in some cases uncontrollable, the dynamics of ξ and $\hat{\xi}$ will differ significantly under high model uncertainty, resulting in unexpected differences between the performance of the real system and the reference model. The adaptation law therefore will become aggressive and the system will possibly be unstable. In order to overcome this issue, in Fig. 2, we propose a control diagram for a new adaptive CBF approach for under-actuated systems, where the internal state ξ from the real system will be combined with the reference output $\hat{\eta}$ to extract a new *modified reference state* $\hat{x}_m$. This modified reference state $\hat{x}_m$ will be used to design the reference controller as well as the dynamics of the reference model instead of using $\hat{x}$ as we did for the fully actuated adaptive CBF.

In the next Section, we will present in more detail the design of the adaptation law as well as formal safety analysis of the proposed L_1 adaptive control Barrier functions.

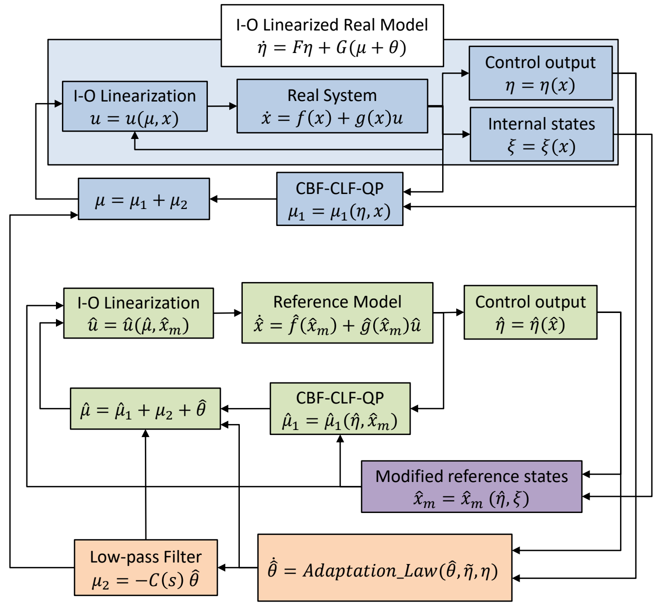


Fig. 2: Control diagram illustrating L_1 adaptive control with a CBF-QP based closed-loop reference model for under-actuated systems. The reference model is modified using the real system's internal states.

C. Control Design and Safety Analysis

Inspired by our prior work on L_1 adaptive CLF for stability [18], in this section, we will present the design of the adaptation law, and then derive stability and safety analysis of the control system. From the Section III, we have the system with uncertainty described by (28) where the nonlinear uncertainty $\theta = \theta(\eta, t)$. As a result, for every time t, we can always find out $\alpha(t), \beta(t) \in \mathbb{R}^m$ such that [8]:

$$\theta(\eta, t) = \alpha(t)||\eta|| + \beta(t). \tag{31}$$

The principle of our method is to design a combined controller $\mu = \mu_1 + \mu_2$, where μ_1 is to control the model to follow the desired reference model and μ_2 is to compensate for the nonlinear uncertainty θ.

In this paper, we present a method to consider a reference model for L_1 adaptive control that arises from CBF-CLF-QP, a safety-critical controller. In particular, we consider the reference model that arises when μ_1 is chosen to be the solution of the QP (23). This reference model is nonlinear and has no closed-form analytical expression.

The state predictor can then be expressed as follows,

$$\dot{\hat{\eta}} = F\hat{\eta} + G\hat{\mu}_1 + G(\mu_2 + \hat{\theta}), \tag{32}$$

where,

$$\hat{\theta} = \hat{\alpha}||\eta|| + \hat{\beta}, \tag{33}$$

and $\hat{\mu}_1$ is computed as the solution of the following CBF-CLF-QP:

CBF-CLF-QP for the state predictor:

$$\hat{\mu}_1^*(\hat{x}) = \underset{\hat{\mu}_1,\hat{\delta}}{\operatorname{argmin}} \quad \hat{\mu}_1^T\hat{\mu}_1 + p\ \hat{\delta}^2 \tag{34}$$

$$\text{s.t.} \quad \dot{V}_\varepsilon(\hat{\eta},\hat{\mu}_1) + \frac{c_3}{\varepsilon}V_\varepsilon(\hat{\eta}) \le \hat{\delta}, \quad \textbf{(CLF)}$$

$$\dot{B}(\hat{x},\hat{u}_1) - \frac{\gamma}{B(\hat{x})} \le 0, \quad \textbf{(CBF)}$$

$$A_c(\hat{x})\hat{u}_1 \le b_c(\hat{x}), \quad \textbf{(Constraints)}$$

$$\hat{u}_1 = \hat{u}_{ff}(\hat{x}) + (L_{\hat{g}}L_{\hat{f}}y(\hat{x}))^{-1}\hat{\mu}_1. \quad \textbf{(IO)}$$

In order to compensate the estimated uncertainty $\hat{\theta}$, we can just simply choose $\mu_2 = -\hat{\theta}$ to obtain

$$\dot{\hat{\eta}} = F\hat{\eta} + G\hat{\mu}_1 \tag{35}$$

which satisfies safety conditions since $\hat{\mu}_1$ is designed through the CBF-CLF-QP in (34). However, $\hat{\theta}$ typically has high-frequency content due to fast estimation. For the reliability of the control scheme and in particular to not violate the unilateral ground force constraints for bipedal walking, it is very important to not have high-frequency content in the control signals. Thus, we apply the L_1 adaptive control scheme to decouple estimation and adaptation [6]. Therefore, we will have

$$\mu_2 = -C(s)\hat{\theta} \tag{36}$$

where $C(s)$ is a low-pass filter with the DC gain being 1.

Define the difference between the real model and the reference model as $\tilde{\eta} = \hat{\eta} - \eta$, we then have,

$$\dot{\tilde{\eta}} = F\tilde{\eta} + G\tilde{\mu}_1 + G(\tilde{\alpha}||\eta|| + \tilde{\beta}), \tag{37}$$

where

$$\tilde{\mu}_1 = \hat{\mu}_1 - \mu_1,\ \tilde{\alpha} = \hat{\alpha} - \alpha,\ \tilde{\beta} = \hat{\beta} - \beta. \tag{38}$$

As a result, we will estimate θ indirectly through α and β, or the values of $\hat{\alpha}$ and $\hat{\beta}$ computed by the following adaptation laws based on the projection operators [13],

$$\begin{aligned} \dot{\hat{\alpha}} &= \Gamma\textbf{Proj}(\hat{\alpha}, y_\alpha), \\ \dot{\hat{\beta}} &= \Gamma\textbf{Proj}(\hat{\beta}, y_\beta), \end{aligned} \tag{39}$$

where Γ is a symmetric positive definite matrix.

We now have the control diagram of L_1 adaptive control Barrier functions described in Fig. 1 for fully-actuated systems and Fig. 2 for under-actuated systems.

In order to find out a suitable function y_α and y_β for the adaptation laws in (39), we will consider the following control Lyapunov candidate function

$$\tilde{V} = \tilde{\eta}^T P_\varepsilon \tilde{\eta} + \tilde{\alpha}^T\Gamma^{-1}\tilde{\alpha} + \tilde{\beta}^T\Gamma^{-1}\tilde{\beta} \tag{40}$$

Let $\delta\mu_1(x) = \mu_1 - K_{lqr}\eta$ where K_{lqr} is the LQR gain such that $A_{cl} = F - GK_{lqr}$ is the solution of the following Lyapunov equation:

$$A_{cl}^T P_\varepsilon + P_\varepsilon A_{cl} + Q = 0. \tag{41}$$

It then implies that

$$\begin{aligned} &(F\tilde{\eta} + G\tilde{\mu}_1)^T P_\varepsilon\tilde{\eta} + \tilde{\eta}^T P_\varepsilon(F\tilde{\eta} + G\tilde{\mu}_1) \\ &= -\tilde{\eta}^T Q\tilde{\eta} + 2\tilde{\eta}^T P_\varepsilon G(\delta\hat{\mu}_1(\hat{x}) - \delta\mu_1(x)). \end{aligned} \tag{42}$$

Because the solution of the QP controllers (23) and (34) are Lipschitz continuous [16], it means that $\delta\mu_1(x)$ is Lipschitz continuous or there will exist a Lipschitz constant $L_{\delta\mu_1}$ such that:

$$||\delta\hat{\mu}_1(x) - \delta\mu_1(x)|| \le L_{\delta\mu_1}||\hat{x} - x||. \tag{43}$$

Furthermore, if $x(\eta,\xi)$ is Lipschitz continuous, we can imply that:

$$||\tilde{x}|| = ||\hat{x}(\hat{\eta},\xi) - x(\eta,\xi)|| \le L_x||\hat{\eta} - \eta||, \tag{44}$$

where L_x is a positive Lipschitz constant of the function $x(\eta,\xi)$.

From (42), (43), (44), it implies that:

$$(F\tilde{\eta} + G\tilde{\mu}_1)^T P_\varepsilon\tilde{\eta} + \tilde{\eta}^T P_\varepsilon(F\tilde{\eta} + G\tilde{\mu}_1) \le -\frac{c_3}{\varepsilon}\tilde{\eta}^T P_\varepsilon\tilde{\eta}. \tag{45}$$

Furthermore, with the property of projection operator [13], we have:

$$\begin{aligned} (\hat{\alpha} - \alpha)^T(\textbf{Proj}(\hat{\alpha}, y_\alpha) - y_\alpha) \le 0, \\ (\hat{\beta} - \beta)^T(\textbf{Proj}(\hat{\beta}, y_\beta) - y_\beta) \le 0. \end{aligned} \tag{46}$$

So, if we choose the projection functions y_α and y_β as,

$$\begin{aligned} y_\alpha &= -G^T P_\varepsilon\tilde{\eta}||\eta||, \\ y_\beta &= -G^T P_\varepsilon\tilde{\eta}, \end{aligned} \tag{47}$$

then from (42), (46), we will have

$$\begin{aligned} \dot{\tilde{V}} + \frac{c_3}{\varepsilon}\tilde{V} \le \frac{c_3}{\varepsilon}\tilde{\alpha}^T\Gamma^{-1}\tilde{\alpha} &+ \frac{c_3}{\varepsilon}\tilde{\beta}^T\Gamma^{-1}\tilde{\beta} \\ &-\tilde{\alpha}^T\Gamma^{-1}\dot{\alpha} - \dot{\alpha}^T\Gamma^{-1}\tilde{\alpha} \\ &-\tilde{\beta}^T\Gamma^{-1}\dot{\beta} - \dot{\beta}^T\Gamma^{-1}\tilde{\beta}. \end{aligned} \tag{48}$$

We assume that the uncertainties α, β and their time derivatives are bounded. Furthermore, the projection operators (39) will also keep $\tilde{\alpha}$ and $\tilde{\beta}$ bounded (see [8] for a detailed proof about these properties.) We define these bounds as follows:

$$\begin{aligned} ||\tilde{\alpha}|| \le \tilde{\alpha}_b, \quad ||\tilde{\beta}|| \le \tilde{\beta}_b, \\ ||\dot{\alpha}|| \le \dot{\alpha}_b, \quad ||\dot{\beta}|| \le \dot{\beta}_b. \end{aligned} \tag{49}$$

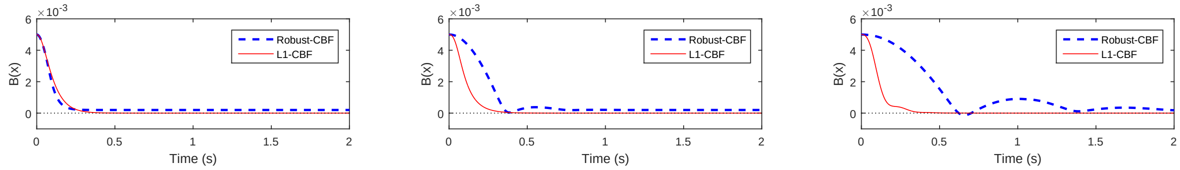
Note that we will not need to specify these bounds in our controller. They are used to prove the stability of our closed-loop system. Combining this with (48), we have,

$$\dot{\tilde{V}} + \frac{c_3}{\varepsilon}\tilde{V} \le \frac{c_3}{\varepsilon}\delta_{\tilde{V}}, \tag{50}$$

where

$$\delta_{\tilde{V}} = 2||\Gamma||^{-1}(\tilde{\alpha}_b^2 + \tilde{\beta}_b^2 + \frac{\varepsilon}{c_3}\tilde{\alpha}_b\dot{\alpha}_b + \frac{\varepsilon}{c_3}\tilde{\beta}_b\dot{\beta}_b). \tag{51}$$

Thus, if $\tilde{V} > \delta_{\tilde{V}}$ then $\dot{\tilde{V}} < 0$. As a result, there exists $T > 0$, so that $\tilde{V}(t) \le \delta_{\tilde{V}}, \forall t \ge T$. In other words, by choosing



(a) No Uncertainty (m=1 kg) (b) m=10 (kg) (c) m=30 (kg)

Fig. 3: Simulation on a single cart system: Comparison of robust CBF and L_1 adaptive CBF with different levels of model uncertainty. Our control goal is to enforce the state-dependent constraint $B(x) \geq 0$. While the robust CBF controller fails to enforce $B(x) \geq 0$ in both (b) and (c), our proposed L_1 adaptive CBF satisfies the constraint for all cases.

the adaptation gain Γ sufficiently large and convergence rate ε sufficiently small, we can drive the Control Lyapunov Function (40) to an arbitrarily small neighborhood $\delta_{\tilde{V}}$ of the origin, implying Input-to-State stability [4]. Therefore the tracking errors between the dynamics model (28) and the reference model (32), $\tilde{\eta}$, and the error between the real and estimated uncertainty, $\tilde{\alpha}$, $\tilde{\beta}$ are bounded as follows:

$$||\tilde{\eta}|| \leq \sqrt{\frac{\delta_{\tilde{V}}}{||P_\varepsilon||}}, ||\tilde{\alpha}|| \leq \sqrt{||\Gamma||\delta_{\tilde{V}}}, ||\tilde{\beta}|| \leq \sqrt{||\Gamma||\delta_{\tilde{V}}}. \quad (52)$$

The convergence properties of the CLF under our controller will therefore contribute to the safety gurantee of the CBF as follows. If the barrier function $B(x)$ is Lipschitz continuous, there exists a Lipschitz constant L_B such that:

$$||\tilde{B}|| = ||\hat{B}(\hat{x}) - B(x)|| \leq L_B||\hat{x} - x|| \leq L_B L_x ||\hat{\eta} - \eta||$$

$$\implies B(x) \geq \hat{B}(\hat{x}) - L_B L_x \sqrt{\frac{\delta_{\tilde{V}}}{||P_\varepsilon||}}. \quad (53)$$

In addition to that, the reference model guarantees safety constraint ($B(\hat{x}) \geq 0$) since it is designed based on a perfect model. Therefore, we have:

$$B(x) \geq -L_B L_x \sqrt{\frac{\delta_{\tilde{V}}}{||P_\varepsilon||}}. \quad (54)$$

This implies that the safety constraint $B(x)$ of the real system will guarantee the *Input-to-State Safety* [12]. In other words, this guarantees B(x) to be lower-bounded by a small negative value. Note that $\delta_{\tilde{V}}$ can be made arbitrarily close to zero by increasing the adaptation gain Γ.

V. Simulation Validation

In this Section, we will present various numerical validation of the proposed method on adaptive control Barrier functions. Following are the systems that we evaluate our proposed controller on:

- Single cart system, a simple linear system,
- Bipedal robot walking on stepping stones while carrying an unknown load, a non-linear under-actuated system.

A. Single Cart System (a simple linear system)

The single cart system has the following linear following dynamics:

$$\ddot{x}_1 = \frac{1}{m}u, \quad (55)$$

where x_1 is the cart position and u in the input force applied to the cart.

In this problem, the control goal is to drive the cart position x_1 as close as possible to the set point of $x_1^d = 1(cm)$, but also guarantees that $x_1 \leq x_1^{max} = 0.5(cm)$. In other words, we enforce a CBF of $B(x) = x_1^{max} - x_1$. We will compare the performance of robust CBF [17] and L1 adaptive CBF. The model uncertainty is introduced by scaling the mass of the cart. As seen in Fig. 3, the result shows that with the mass scale of 30, the L1 adaptive CBF still guarantees the safety constraint while the robust CBF controller fails to keep the system state in the safety set even with the mass scale of 10.

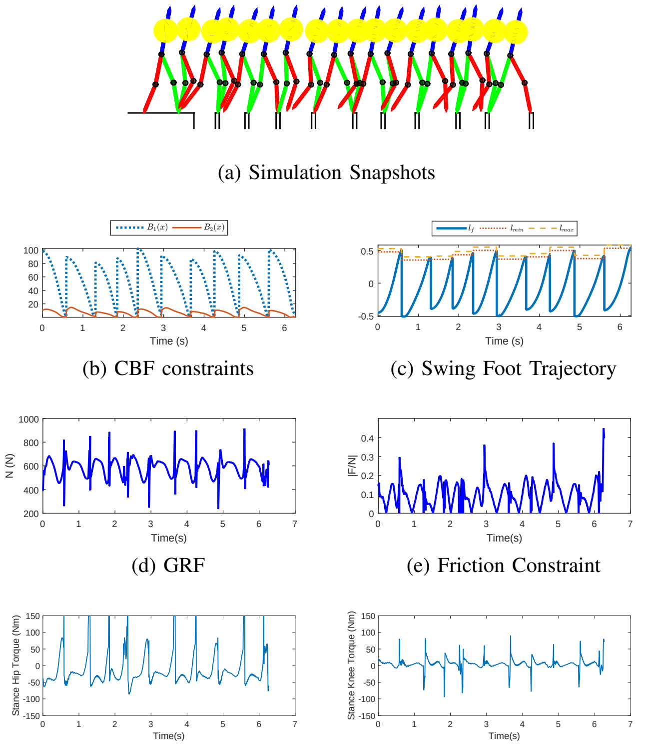
B. Bipedal robot walking on stepping stones while carrying an unknown load (a non-linear under-actuated system)

In this part, we will apply the proposed L_1 adaptive CBF method for the problem of bipedal robotic walking on stepping stones while carrying an unknown load. For this problem, the robot needs to address model uncertainty arising from an unknown load carried on the torso, while enforcing safety-critical constraints to enforce precise footstep placement. Details about the formulation of CBF and control parameters for this application can be found in [20]. A similar numerical study for this application was also conducted in our prior work on robust CBF [17], which was successfully validated with an unknown load of 15 kg (47 $\%$ of the robot weight). With this development of L_1 adaptive control, the robot can even carry 30 kg of unknown load, which is 94 $\%$ of the robot weight while adjusting step length for every walking step within the range of $[35:55]$ cm (see Fig. 4). Note that the Robust CBF controller [17] fails right at the first walking step with this high level of model uncertainty.

In Fig. 4, we include the snapshot of the simulation, CBF constraints, swing foot trajectory, ground reaction force and friction constraint, showing that the control goal and physical constraints are met. The plots of the stance leg's joint torques also show the smoothness of the control inputs.

VI. Conclusion

We have presented a novel method of L_1 adaptive safety-critical control that enables the ability to enforce stability and safety-critical constraints under high level of model uncertainty for both fully-actuated and under-actuated systems. The proposed adaptive control framework uses the nonlinear



(a) Simulation Snapshots

(b) CBF constraints (c) Swing Foot Trajectory

(d) GRF (e) Friction Constraint

(f) Stance Hip Torque (g) Stance Knee Torque

Fig. 4: Simulation of L_1 adaptive CBF controller on a bipedal robot walking on stepping stones while carrying an unknown load of 30 kg (94 % of the robot weight).

closed-loop reference model that is controlled by CBF-CLF-QP. Our method can be applied for both fully-actuated and under-actuated systems. We validate our approach in simulation for a single cart system, and bipedal robotic walking. In comparison with the robust CBF presented in [17], the L_1 adaptive CBF controller has two advantages. Firstly, since it can estimate and adapt to model uncertainty, it maintains consistent performance under different level of model uncertainty. The robust CBF controller instead tends to be aggressive even without model uncertainty. Secondly, in the same problem setup, the L_1 adaptive CBF controller can address larger model uncertainty. For example, for the problem of bipedal robot walking on stepping stones while carrying an unknown load, the adaptive controller works with a load up to 30 kg but the maximum load that the robust one can handle is 15 kg.

REFERENCES

[1] A. D. Ames, K. Galloway, K. Sreenath, and J. W. Grizzle, "Rapidly exponentially stabilizing control lyapunov functions and hybrid zero dynamics," *IEEE Transactions on Automatic Control (TAC)*, vol. 59, no. 4, pp. 876–891, April 2014.

[2] A. D. Ames, J. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs with application to adaptive cruise control," in *IEEE Conference on Decision and Control*, 2014, pp. 6271–6278.

[3] A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, "Control barrier functions: Theory and applications," in *2019 18th European Control Conference (ECC)*. IEEE, 2019, pp. 3420–3431.

[4] D. Angeli, E. Sontag, and Y. Wang, "A characterization of integral input-to-state stability," *IEEE Transactions on Automatic Control*, vol. 45, pp. 1082–1097, 2000.

[5] C. Cao and N. Hovakimyan, "Stability margins of l1 adaptive controller: Part ii," *Proceedings of American Control Conference*, pp. 3931–3936, 2007.

[6] ——, "Design and analysis of a novel l1 adaptive controller with guaranteed transient performance," *IEEE Transactions on Automatic Control*, vol. 53, no.2, pp. 586–591, March, 2008.

[7] C.Cao and N. Hovakimyan, "L1 adaptive output-feedback controllers for non-strictly-positive-real reference systems: Missile longitudinal autopilot design." *Journal of Guidance, Control, and Dynamics*, vol. 32, pp. 717–726, 2009.

[8] C.Cao and N.Hovakimyan, "L1 adaptive controller for a class of systems with unknown nonlinearities: Part i," *American Control Conference, Seattle, WA*, pp. 4093–4098, 2008.

[9] D. D. Fan, J. Nguyen, R. Thakker, N. Alatur, A.-a. Agha-mohammadi, and E. A. Theodorou, "Bayesian learning-based adaptive control for safety critical systems," in *2020 IEEE International Conference on Robotics and Automation (ICRA)*. IEEE, 2020, pp. 4093–4099.

[10] K. Galloway, K. Sreenath, A. D. Ames, and J. W. Grizzle, "Torque saturation in bipedal robotic walking through control lyapunov function based quadratic programs," *IEEE Access*, vol. PP, no. 99, p. 1, April 2015.

[11] S.-C. Hsu, X. Xu, and A. D. Ames, "Control barrier function based quadratic programs with application to bipedal robotic walking," in *American Control Conference*, 2015, pp. 4542–4548.

[12] S. Kolathaya and A. D. Ames, "Input-to-state safety with control barrier functions," *IEEE Control Systems Letters*, vol. 3, no. 1, pp. 108–113, 2019.

[13] E. Lavretsky and T. E. Gibson, "Projection operator in adaptive systems," *arXiv preprint arXiv:1112.4232*, 2011.

[14] B. T. Lopez, J.-J. E. Slotine, and J. P. How, "Robust adaptive control barrier functions: An adaptive and data-driven approach to safety," *IEEE Control Systems Letters*, vol. 5, no. 3, pp. 1031–1036, 2020.

[15] A. Mehra, W.-L. Ma, F. Berg, P. Tabuada, J. W. Grizzle, and A. D. Ames, "Adaptive cruise control: Experimental validation of advanced controllers on scale-model cars," in *American Control Conference*, 2015, pp. 1411–1418.

[16] B. Morris, M. J. Powell, and A. D. Ames, "Sufficient conditions for the Lipschitz continuity of QP-based multi-objective control of humanoid robots," *Proc. 52nd IEEE Conf. Decision and Control*, pp. 2920–2926, 2013.

[17] Q. Nguyen and K. Sreenath, "Robust safety-critical control for dynamic robotics," *IEEE Transactions on Automatic Control*, pp. 1–1, 2021.

[18] Q. Nguyen and K. Sreenath, "L1 adaptive control for bipedal robots with control lyapunov function based quadratic programs," in *American Control Conference*, 2015, pp. 862–867.

[19] ——, "Optimal robust control for bipedal robots through control lyapunov function based quadratic programs," in *Robotics: Science and Systems*, 2015.

[20] ——, "Safety-critical control for dynamical bipedal walking with precise footstep placement," in *The IFAC Conference on Analysis and Design of Hybrid Systems*, 2015, pp. 147–154.

[21] ——, "Exponential control barrier functions for enforcing high relative-degree safety-critical constraints," in *American Control Conference*, 2016.

[22] V. V. Patel, C. Cao, N. Hovakimyan, K. A. Wise, and E. Lavretsky, "L1 adaptive controller for tailless unstable aircraft in the presence of unknown actuator failures," *International Journal of Control*, vol. 82, pp. 705–720, 2009.

[23] A. J. Taylor and A. D. Ames, "Adaptive safety with control barrier functions," in *2020 American Control Conference (ACC)*. IEEE, 2020, pp. 1399–1405.

[24] G. Wu and K. Sreenath, "Safety-critical and constrained geometric control synthesis using control lyapunov and control barrier functions for systems evolving on manifolds," in *American Control Conference*, 2015, pp. 2038–2044.

[25] X. Xu, P. Tabuada, J. W. Grizzle, and A. D. Ames, "Robustness of control barrier functions for safety critical control," in *The IFAC Conference on Analysis and Design of Hybrid Systems*, 2015, pp. 54–61.