Safety-Critical Control of a Planar Quadrotor

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***Abstract*— Aerial robots such as quadrotors are subject to various constraints such as connectivity and collision constraints, that have to be strictly enforced for safe operation. Such constraints are dynamic constraints and require the state of the system to be forward-invariant with respect to a safe set. We propose a control design method for a planar quadrotor subject to these safety constraints. The proposed controller explicitly considers the nonlinear underactuated dynamics of a planar quadrotor and formulates a sequential quadratic program that uses control Lyapunov functions to guarantee stability and control Barrier functions to guarantee safety. We introduce and construct an augmented control Barrier function that enforces a safety region in position space while explicitly taking into account the quadrotor's orientation. We present several numerical results to validate the proposed controller, including (a) setpoint regulation subject to a single static obstacle, (b) trajectory tracking with multiple static obstacles, and (c) trajectory tracking with a dynamic obstacle.**

I. INTRODUCTION

Aerial robots have become a good platform for control and robotics research, not only because of the existence of many well-designed quadrotors ranging in size from centimeters to meters with payloads up to a few kilograms, but also for the potential applications they have enabled, such as persistence surveillance, sensor network-based monitoring, aerial object retrieval, aerial transportation, etc., see [5], [8], [10]–[12], [15], [16], [21], [25]. Moreover, studying the complex dynamics and constrained control of quadrotors could deepen our understanding of underactuated systems, especially those in aerospace applications.

In order to take advantage of the capabilities of an aerial robot, the on board controller needs to take into account various constraints [3], such as range, power, communication, and safety constraints. Several approaches have been developed to study constrained control problem for different systems. Model Predictive Control (MPC) [14] employs control input as the solution of a state-dependent and finite-horizon optimization problem, wherein all the constraints are treated as the constraints of this optimization directly. Although MPC is handy in addressing different types of constraints, to guarantee stability of MPC controller requires proper tuning. In most case, to apply MPC in real time requires considering the types of systems and constraints so that we could simplify the general optimization into a convex optimization [19], [26]. For safety constraints, a variety of methods exist. One approach is to adjust the reference command using a pre-filter called *reference governor* to enforce safety constraints, [4], [9]. Other methods extend optimal control method to systems under uncertainty, where the control process becomes a *differential game* [7], [17].The common idea these research works share is to identify a suitable control function such that the forward invariances of some sets are guaranteed, where we call these sets safety sets.

To better analyze the properties of safety sets, recent work has developed the concept of Barrier Function (BF) [22]–[24], [27]. The concept of "Barrier Function" is an analogy taken from optimization, which guarantees the closed loop system trajectory never escapes the safety sets. Detailed construction of barrier functions for nonlinear systems is presented in [22], [24] for constraints on the full and partial states. A converse barrier certificate theorem, similar to the converse Lyapunov function theorems in stability theory, is proposed in [27]. Compared to these works with explicit feedback laws, Control Barrier Function (CBF) [2] is introduced to contain the control input directly. By converting the safety constraints into state-dependent linear inequality constraints on the inputs, Ames et al. are able to propose a controller for adaptive cruise control based on online QP. This approach combines control Lyapunov functions (CLFs) for stability [1] and CBFs for safety and solves a state-dependent quadratic program (QP) pointwise in time for the control input [6]. The concept of CBF has also been extended to fully actuated, simple mechanical systems on Riemannian manifolds [28], with the same control architecture.

Our goal in this paper is to develop a safety critical control for the planar quadrotor using the concept of CLF and CBF. This is challenging due to underactuated nature of the quadrotor dynamics and the fact that the quadrotor's translational part depends on its orientation. The contributions of our work with respect to prior work [2], [6], [28] are shown below:

- We propose an augmented CBF candidate for the planar quadrotor to establish the forward-invariance of a safety set in position space while explicitly taking into account the quadrotor's orientation. We also show this CBF candidate is an actual CBF.
- We develop a sequential QP control scheme based on this CBF and a virtual CLF to address the safety constraints of a planar quadrotor.
- We demonstrate the performance of this controller through numerical simulations with single and multiple static obstacles as well as dynamic obstacles.

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The rest of the paper is organized as follows: Section II provides a detailed discussion about the concepts of CLFs and CBFs; Section III presents the system dynamics of a planar quadrotor and formulates the constrained control problem. Section IV gives a detailed construction of an augmented CBF. Section V proposes the sequential QP control design methods. Section VI presents numerical results for tracking both static and dynamic references, and discusses the advantages and limitations of the proposed control method.

II. Mathematical Preliminary

This section introduces the concepts of CLFs and CBFs. We extend results in [2] to propose a CBF which can handle time varying constraints. For a more detailed discussion on CLFs and CBFs, we refer the reader to [2], [28].

A. Exponentially-Stabilizing Control Lyapunov Function

Consider a control affine system shown as,

$$\dot{x} = f(x) + g(x)u, \qquad x(t_0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Then, for the above system, a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is called an Exponentially Stabilizing Lyapunov Function (ES-CLF) if there exist constants $c_1, c_2, c_3 > 0$ such that,

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2, \ \inf_{u\in\mathbb{R}^m}\{L_fV + L_gVu + c_3V\} \le 0.$$

***Remark** 1:* ES-CLFs give a qualitative analysis of the stability of the origin. If such a function exists, we can determine a feedback law based on V so that the corresponding closed-loop system is ensured to have *exponential stability.* A key fact about CLFs is its implicit dependence on time. Recall that in stability theory, explicit dependence of the Lyapunov function on time requires additional conditions to determine the asymptotic behavior of the system's trajectory.

B. Control Barrier Function (CBF)

As introduced in [2], a CBF can be constructed based on a safe set in the state space. For system (1), suppose we have a continuously differentiable function $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ with its super level set $\mathcal{C} = \{x \in \mathbb{R}^n \mid h(t,x) \ge 0\}$. If this set admits a non-empty interior, denoted as $\mathcal{C}^\circ$, for all time $t \in [0,\infty)$, then a smooth scalar function $B : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is called a CBF of $\mathcal{C}$ if there exist two class $\mathcal{K}$ functions α_1, α_2 and $\eta > 0$ such that

$$\frac{1}{\alpha_1(h(t,x))} \le B(h) \le \frac{1}{\alpha_2(h(t,x))},$$
$$\inf_{u\in\mathbb{R}^m}\{B'(h)(\frac{\partial h}{\partial t} + L_fh + L_gh\cdot u) - \frac{\eta}{B}\} \le 0,$$

for all $t \in [0,\infty)$ and $x \in \mathcal{C}^\circ$.

***Remark** 2:* The idea of CBFs stem from dynamic system theory. According to [2], to stay safe, the system trajectory should always remain within the safe set defined by h. This is equivalent to the fact that these sets are *forward invariant* with respect to the closed loop system. Thus by enforcing the CBF condition, the value of h would always remain nonnegative, and thus guarantee safety, i.e $x(t) \in \mathcal{C}$.

***Remark** 3:* Note that our definition of CBF is dependent on the scalar set function h, which can be potentially time-varying.

***Lemma** 1: (Safety guarantee of CBF):* For system (1), if there exists a control input u satisfying $B'(h)(\frac{\partial h}{\partial t} + L_fh + L_gh\cdot u) - \frac{\eta}{B} \le 0$, where $\eta > 0$ and $t \in [0,\infty)$, then the set $\mathcal{C} = \{x \in \mathbb{R}^n \mid h(t,x) \ge 0\}$ is forward invariant, equivalently, $x(0) \in \mathcal{C} \implies x(t) \in \mathcal{C}$.

Proof: The proof follows from [2, Lemma 1], which guarantees that the set $\{x \mid h(x) \ge 0\}$ is forward invariant for system (1) whenever there exists a control u satisfying $\dot{B} = B'(h)(L_fh + L_gh\cdot u) \le -\frac{\eta}{B}$. Since in our case h is explicitly dependent on time, applying Chain rule to obtain $\dot{B}$ then yields the result. ■

III. Constrained Control Problem for Planar Quadrotor

Consider the system shown in Fig. 1 where a quadrotor is confined to move within a plane. Its dynamics is given below:

$$\begin{aligned} m\ddot{x} &= F\sin\theta, \\ m\ddot{y} &= F\cos\theta - mg, \\ J\ddot{\theta} &= -M, \end{aligned} \tag{2}$$

where g is the acceleration due to gravity, m, J are the mass and inertia of the planar quadrotor, and F, M are scalar inputs representing the thrust and moment applied by the propellers. These dynamics can be written in state-space as

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\mathbf{u},$$

where $\mathbf{x} = \begin{bmatrix} x & y & \theta & \dot{x} & \dot{y} & \dot{\theta}\end{bmatrix}^T$ and $\mathbf{u} = \begin{bmatrix} F & M\end{bmatrix}^T$ are the state and control input respectively.

Constrained control problem for a planar quadrotor: Consider the system (2) with $\mathbf{x}(t), \mathbf{u}(t)$. Assume we are given a smooth reference trajectory $(x_d(t), y_d(t), \theta_d(t))$ for the quadrotor to track, along with a list of potentially time-varying safe sets $\mathcal{C}_i = \{(x,y) : g_i(t,x,y) \ge 0\}$, as determined by the functions $g_i(t,x,y) = (x - x_i(t))^2 + (y - y_i(t))^2 - b_i(t)^2,\ b_i \ge 0, i = 1,2,\cdots,k$. Define the overall safe set in state-space as $\mathcal{C} = \cap_{i=1}^k \mathcal{C}_i$, and assume that the interior $\mathcal{C}^\circ$ is nonempty. The control goal then is to design a feedback law for $F, M : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}$ such that it satisfies the following:

$(x,y) \to (x_d, y_d) \in \mathcal{C}$, as $t \to \infty$	(Position tracking)
$0 \le F \le F_{\max}, \quad \lvert M\rvert \le M_{\max}$	(Input saturation)
$(x(t), y(t)) \in \mathcal{C}, \forall t \in [0,\infty)$	(Safety constraint)

As can be seen in Fig. 1, the obstacle's shape is approximated as a union of several circles with center (x_i, y_i), with the exteriors of the circles representing the safe regions $\mathcal{C}_i$. As long as the reference trajectory belongs to the overall safe region, $\mathcal{C}$, asymptotic stability should be attained.

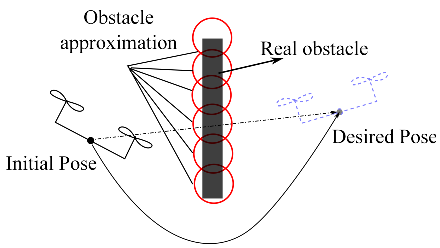

Fig. 1: Constrained control problem of planar quadrotor. A reference trajectory (straight line) and a list of safe sets are provided (exterior of the red circles). The overall safe region is the intersection of these safe sets, which serves as an approximation of the real obstacle (solid black rectangle). The control goal is to track the reference trajectory while simultaneously strictly enforcing that the state remains in the safe region.

IV. Augmented CBF Construction for a Planar Quadrotor

For fully-actuated simple mechanical systems with position-based constraints, a general construction of CBFs can be found in [28]. The process is to expand the safety region, specified in position space, to the whole state-space as follows:

$$h_i(x, y, \dot{x}, \dot{y}) := \gamma_i \alpha_i(g_i(x, y)) + \dot{g}_i(x, y),$$

where $\gamma_i > 0$ and α_i is a class $\mathcal{K}$ function. Then, as shown in [28, Prop. 1], guaranteeing the state constraint $h_i(x, y, \dot{x}, \dot{y}) \geq 0$ implies the position constraint $g_i(x, y) \geq 0$.

However, for (2), this method does not work. The major challenge lies in the fact that the derivatives of the normal CBF are not dependent on all the control inputs, or equivalently, the *lack of a well-defined vector relative degree.* For instance, as discussed in Section II-B, we can select a candidate

$$B_i(\mathbf{x}) = \frac{1}{h_i(x, y, \dot{x}, \dot{y})},$$

and check whether

$$\inf_{\mathbf{u}\in U} \{L_f B(\mathbf{x}) + L_g B(\mathbf{x})\mathbf{u} - \frac{\eta}{B(\mathbf{x})}\} \leq 0. \tag{3}$$

holds for all $\mathbf{x}$.

For this case, $L_g B$ has the expression $[\bullet, 0]$. Then (3) depends only on the thrust part of input $u = \begin{bmatrix} F & M \end{bmatrix}^T$, which can only be nonnegative. Thus, for some cases, there may not exist a feasible $\mathbf{u} \in U$ such that (3) is satisfied.

The key idea to tackle this difficulty is to augment the original position region function g_i by adding another term that is dependent on the orientation θ. This idea comes from the fact that *the capability of a quadrotor to move away from an obstacle* varies as its orientation changes. For example, as shown in Fig. 2, the red circle represents an obstacle. It is obvious that the quadrotor in blue dashed line is more capable of avoiding obstacle than the one in black solid line, since the quadrotor thrust direction is fixed and the thrust magnitude can only be positive. Hence intuitively, the normal CBF can work for the former case but probably would fail for the latter, since positive thrust would bring the quadrotor closer to the obstacle.

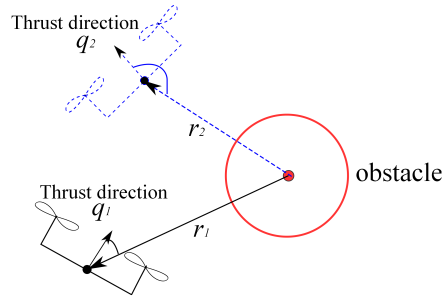

Fig. 2: Diagram showing the motivation of augmented CBF: black solid quadrotor is less capable than the blue dashed one to avoid the red circled obstacle since $q_1 \cdot r_1 < 0 < q_2 \cdot r_2$.

To address this, our CBF-based controller should be able to adjust the orientation of the quadrotor, and thus the moment should be included in the derivatives of the CBF. Based on this argument, we propose a detailed construction of an *augmented CBF* for (2). We begin by considering the safety region defined by $\mathcal{C}_0 = \{(x, y) : (x - x_o)^2 + (y - y_o)^2 \geq b^2\}$. First, we reduce the size of the original safe region by a factor β, resulting in,

$$g(x, y) = (x - x_o)^2 + (y - y_o)^2 - \beta b^2, \ \beta > 1,$$

where x_o, y_o, b are smooth time-varying functions. We then augment the value of b based on the current orientation as following:

$$\hat{g}(x, y, \theta) := g(x, y) - \sigma(s),$$

where $s := \sin\theta(x - x_o) + \cos\theta(y - y_o)$ and the properties of the smooth function $\sigma : \mathbb{R} \to \mathbb{R}$ would be determined later.

Remark 4: For convenience, we define the direction of thrust F as $q = (\sin\theta, \cos\theta)$ and the distance vector from the obstacle center to the current location of the quadrotor as $r = (x - x_o, y - y_o)$. Then the argument s in $\sigma(s)$ equals $r \cdot q$, as shown in Fig. 2, and can be treated as a *signed measure of the quadrotor's ability to escape from the obstacle from the current pose.*

Next following the same construction procedure in [28], we can expand the safety set to the whole state-space:

$$\hat{h} := \gamma\alpha(\hat{g}) + \dot{\hat{g}} = \gamma\alpha(\hat{g}) + \dot{g}(x, y) - \sigma'(s)(p\dot{\theta} + v),$$

where $p := \cos\theta(x - x_o) - \sin\theta(y - y_o) = r \times q$ and $v := \sin\theta(\dot{x} - \dot{x}_o) + \cos\theta(\dot{y} - \dot{y}_o)$ and $\alpha : \mathbb{R} \to \mathbb{R}$ is a class $\mathcal{K}$ function.

To simplify the computation of $\dot{\hat{h}}$, first express the derivative of p, r as:

$$\dot{p} = -s\dot{\theta} + w, \quad \dot{v} = q\dot{\theta} + \sin\theta(\ddot{x} - \ddot{x}_o) + \cos\theta(\ddot{y} - \ddot{y}_o), \tag{4}$$

with $w := \cos\theta(\dot{x} - \dot{x}_o) - \sin\theta(\dot{y} - \dot{y}_o)$.

Substituting (4) into $\dot{\hat{h}}$ yields

$$\dot{\hat{h}} = \gamma\alpha'(\hat{g})\cdot\dot{\hat{g}} - \sigma''(s)(p\dot{\theta}+v)^2 - \sigma'(s)\dot{\theta}(2w - s\dot{\theta}) + \sigma'(s)(\sin\theta\ddot{x}_o + \cos\theta\ddot{y}_o) - p\sigma'(s)\ddot{\theta} - \sigma'(s)(\sin\theta\ddot{x} + \cos\theta\ddot{y}) + \ddot{g}(x,y).$$

Plugging the system dynamics (2) into $\dot{\hat{h}}$ yields

$$\dot{\hat{h}} = \frac{p\sigma'(s)}{J}M + \frac{1}{m}(2s - \sigma'(s))F + C.$$

where F, M are the thrust and moment part of u and the term C is independent of u.

A candidate CBF $B = 1/\hat{h}^a$ with $a > 0$ is selected and we have the following lemma:

***Lemma** 2: (Safety guarantee of augmented CBF):* Suppose the scalar function $\sigma(s)$ satisfies the following properties:

$$\sigma'(s) < 0, \qquad \text{(Strictly decreasing)}$$
$$\nu_- = \lim_{s\to\infty}\sigma(s) < 0 < \lim_{s\to-\infty}\sigma(s) = \nu_+, \qquad \text{(Global boundedness)}$$
$$2s - \sigma'(s) > 0, \quad \forall s \in (-\sqrt{\beta}b, 0), \qquad \text{(Bounded derivative)}$$

with the values $|\nu_-|, \nu_+ < (\beta-1)b^2$. Then the candidate function B is an almost global CBF for (2). We can *guarantee safety* for the trajectory of (2), provided that $\hat{h}\big|_{t=0} \geq 0$ and $F_{\max}, M_{\max}$ are large enough.

Proof: Assuming that the thresholds $F_{\max}, M_{\max}$ are unbounded, it holds that

$$\dot{B} - \frac{\gamma}{B} \leq 0 \Leftrightarrow -\frac{1}{\hat{h}^{a+1}}(\dot{\hat{h}} + \gamma\hat{h}^{2a+1}) \leq 0, \quad \Leftrightarrow \dot{\hat{h}} + \gamma\hat{h}^{2a+1} \geq 0. \tag{5}$$

Substituting the expression of $\dot{\hat{h}}$ yields

$$\frac{p\sigma'(s)}{J}M + \frac{1}{m}(2s - \sigma'(s))F \geq -C + \gamma\hat{h}^{2a+1}. \tag{6}$$

If $p \neq 0$, then $p\sigma'(s) \neq 0$ which means that we could always select a moment to satisfy (6) with:

$$M = \frac{-C + \gamma\hat{h}^{2a+1}}{p\sigma'(s)}, \quad F = 0$$

If $p = 0$, then by definition it follows that r, q are parallel to each other. For the case when their direction coincides, then $s = r\cdot q > 0$ which implies $2s - \sigma'(s) > 0$ since $\sigma'(s) < 0$ for $s > 0$. Hence, we can use a large enough thrust to satisfy (5). Otherwise for the case $p = 0$, the only condition when (5) may fail is that $s \leq -\sqrt{\beta}b$ by the third condition. This is actually two-dimensional compact manifold in $\mathbb{R}^2 \times \mathbb{S}^1$ and thus has Lebesgue measure zero in the state space. So B is an almost global CBF. For the case when B fails, we have that $s \leq -\sqrt{\beta}b$ which means that $|s| = \sqrt{(x-x_o)^2 + (y-y_o)^2} \geq \sqrt{\beta}b$ which implies that $(x-x_o)^2 + (y-y_o)^2 \geq \beta b^2 > b^2$. This implies that the trajectory would be safe for the cases when B fails to work. When the condition of B is satisfied, by *Lemma*. 1, the system trajectory would always remain within the region $\mathcal{C}_1 = \{(x,y) : (x-x_o)^2 + (y-y_o)^2 \geq \beta b^2 + \sigma(s)\} \subset \{(x,y) : (x-x_o)^2 + (y-y_o)^2 \geq (\beta b^2 + (1-\beta)b^2\} = \{(x,y) : (x-x_o)^2 + (y-y_o)^2 \geq b^2\} = \mathcal{C}_0$ since $|\sigma|$ is bounded by $(\beta-1)b^2$. Hence, the system trajectory will always remain within $\mathcal{C}$, and thus remain safe.

■

***Remark** 5:* The role of the scalar function $\sigma(s)$ is to adjust the radius b based on *the ability to escape from the obstacle center.* Some candidate functions that satisfy the conditions of Lemma 2 are,

$$\sigma_1(s) = k_1\frac{\exp(-k_2 s + k_3) - 1}{\exp(-k_2 s + k_3) + 1}, \quad \sigma_2(s) = -a_1\arctan(a_2 s + a_3). \tag{7}$$

where the constants $k_1, k_2, k_3, a_1, a_2, a_3$ can be tuned to satisfy the requirements.

V. Sequential QP Control Design for a Planar Quadrotor

Based on the augmented CBF constructed in the previous Section, we propose a sequential optimization scheme for control design. The underlying idea is inspired by the backstepping method in geometric control [11], [12], [21], which separate the fast orientation dynamics from the slow translational dynamics. Similar to this, the controller proposed comprises of two levels: the *position level QP* and the *orientation level QP*, as will be discussed in detail.

The first step of our control method is to construct a Barrier function B_i for each safety region $\mathcal{C}_i$. Then we make an assumption that the underactuated part is "fully-actuated" with the virtual dynamics:

$$m\ddot{x} = f_{v1}, \quad m\ddot{y} = f_{v2}.$$

Next, select a quadratic CLF for this virtual system as:

$$\hat{V} = \frac{1}{2}m e_v\cdot e_v + \frac{1}{2}k_1 e_x\cdot e_x + \varepsilon_1 e_x\cdot e_v$$

where $e_x = [x - x_d, y - y_d]^T, e_v = [\dot{x} - \dot{x}_d, \dot{y} - \dot{y}_d]^T$, with $k_1, \varepsilon_1 > 0$ selected to make $\hat{V}$ quadratic. We can then compute a virtual force based on $\hat{V}$ through the QP:

Position Level QP (virtual force computation)

$$f_v = \underset{f\in\mathbb{R}}{\text{argmin}}\ \frac{1}{2}f^TQf \quad \text{subject to } \dot{\hat{V}}(x,y,f) + \eta_1\hat{V}(x,y) \leq 0,$$

where $\eta_1 > 0$. The solution f_v is computed as a *virtual force* and passed onto the lower orientation level of optimization as an input parameter.

Based on this input f_v, we further compute a desired thrust $F_c = \max\{f_v\cdot[\sin\theta, \cos\theta]^T, 0\}$ and a desired angle $\theta_c = \arctan(f_{v1}/f_{v2})$. In this way, we can construct a CLF for the orientation part only:

$$V_\theta = \frac{1}{2}J(\dot{\theta} - \dot{\theta}_c)^2 + \frac{1}{2}k_2(\theta - \theta_c)^2 + \epsilon_2(\theta - \theta_c)(\dot{\theta} - \dot{\theta}_c). \tag{8}$$

Then, the orientation level QP is constructed to obtain our actual control inputs F and M:

Orientation Level QP (virtual force tracking and safety):

$$[F^*, M^*] = \underset{F,M \in \mathbb{R}}{\operatorname{argmin}} \ \frac{1}{2}\lambda_1 (F - F_c)^2 + \frac{1}{2}(M)^2 + \frac{1}{2}\lambda_2 \delta^2$$
$$\text{subject to} \quad \dot{V}_\theta(M) + \eta_2 V_\theta \le \delta$$
$$\dot{B}_j(F,M) \le \frac{\gamma_j}{B_j}, \ j = 1,2,\cdots,k,$$
$$0 \le F \le F_{\max}, \ |M| \le M_{\max}$$

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where $\dot{B}_j(F,M)$ is as computed in (5) and $\lambda_1, \lambda_2, \eta_2, \gamma_j$ are all strictly positive. Regarding the safety property of this controller, we have the following proposition:

***Proposition** 1: (Safety Guarantee of Sequential QP-Controller):* If the following assumptions are satisfied:

- Suppose at each time $t \in [0,\infty)$, the sequential QP controller always has a solution F, M.
- The initial state of (2) satisfies $0 < B_j|_{t=0} < \infty$ for all $j = 1,2,\cdots,k$.

then the system trajectory would always remain within $\mathcal{C}$.

Proof: We can draw the conclusion based on *Lemma*. 2. Since for every $j = 1,2,\cdots,k$, the value $B_j|_{t=0}$ remains positive and bounded. This is equivalent to $h_j|_{t=0} \ge 0$. Also, the first assumption guarantees that the condition of every CBF is enforced and thus $h_j(t,\mathbf{x}) \ge 0$ for all $t > 0$. Thus, the system trajectory would remain within every region $\mathcal{C}_j$ and thus their intersection $\mathcal{C}$. ■

***Remark** 6:* Sequential QP is an interesting topic in constrained nonlinear optimization research [20]. Theorems in [13] can guarantee the existence and uniqueness of the solution under certain conditions. According to [18], the solution of a single QP is Lipschitz continuous under proper conditions. However, for our controller, we can only expect piecewise continuity due to the presence of truncation on F_c.

VI. Simulation Result of a Planar Quadrotor

In this section, we numerically validate the proposed Sequential-CBF-QP controller on a planar quadrotor. The parameters of this planar quadrotor are given as: $m_q = 1.0(kg)$, $J_q = 0.25(kg \cdot m^2)$. We pick a sigmoid function for $\sigma(s)$ for constructing the augmented CBF, shown in (7). Simulations are performed in Matlab 2015b using the rigid ode integrator "ode15s". The online QP is solved using "Interior Point method" with tolerance 1×10^{-6} for the first level and 1×10^{-9} for the second level. We show the results of three scenarios with some detailed discussions.

A. Setpoint Regulation with a Single Static Obstacle

The first scenario is to regulate to a hover setpoint at the position $(6,6)$ subject to a safety constraint imposed by a single large obstacle. We test two controllers: a Sequential-QP controller for the quadrotor that does not explicitly account for CBF based on constraints, and the Sequential-CBF-QP controller. The input saturations are $F_{\max} = 40N$, $M_{\max} = 20N \cdot m$. The quadrotor starts at the initial configuration $(x_0, y_0, \theta_0) = (0, 0, -\pi/2)$ with zero initial velocity. Both controllers are subject to the same input saturation constraints.

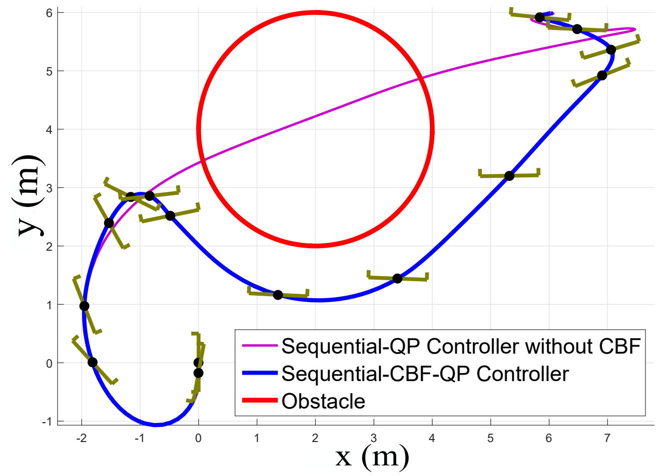

Fig. 3: Setpoint regulation with a single static obstacle: Quadrotor position trajectories are shown for two controllers. The control goal is to perform static setpoint regulation, i.e., go from a given initial state to a desired final state. As seen clearly, the Sequential-QP controller without CBF violates the safety-critical constraint and enters the unsafe region denoted by the red obstacle, while the Sequential-CBF-QP controller strictly enforces the safety constraint. Simulation video: `http://youtu.be/XmPmyklZUmI`.

The resulting motions for the two controllers are compared in Fig. 3. As can be seen, the Sequential-QP controller without CBF results in violation of the safety constraint whereas our proposed Sequential-CBF-QP controller is able to strictly enforce the safety constraint and keep the quadrotor state within the safe set. The figure illustrates snapshots of the quadrotor to depict changes in the orientation of the quadrotor as it moves along the trajectory. Fig. 4 depicts thrust and moment inputs generated by the controller. As can be seen, our proposed controller is able to strictly enforce the input saturation constraints. The inputs are piecewise continuous. Fig. 5 plots out the error trajectory, illustrating the asymptotic stability properties of the controller.

B. Trajectory Tracking with Multiple Static Obstacles

In the second scenario, the quadrotor is commanded to track a dynamic time-varying trajectory subject to multiple static obstacles. The obstacles approximate the shape of a rectangle and are specifically designed to intersect the reference trajectory, i.e., the desired reference directly passes through the obstacles. The main purpose of this trial is to illustrate the strict safety guarantees of this controller, where the controller relaxes enforcing stable trajectory tracking when its no longer feasible to guarantee safety. The reference trajectory is given by $(x_d, y_d) = [4\cos(t), 2.5\cos(0.5t + \pi/4)]$, the quadrotor starts at an initial configuration $(x_0, y_0, \theta_0) = [0, -3, \pi/2]$ and with initial velocity $(\dot{x}_0, \dot{y}_0, \dot{\theta}_0) = [2, 0, 0]$.

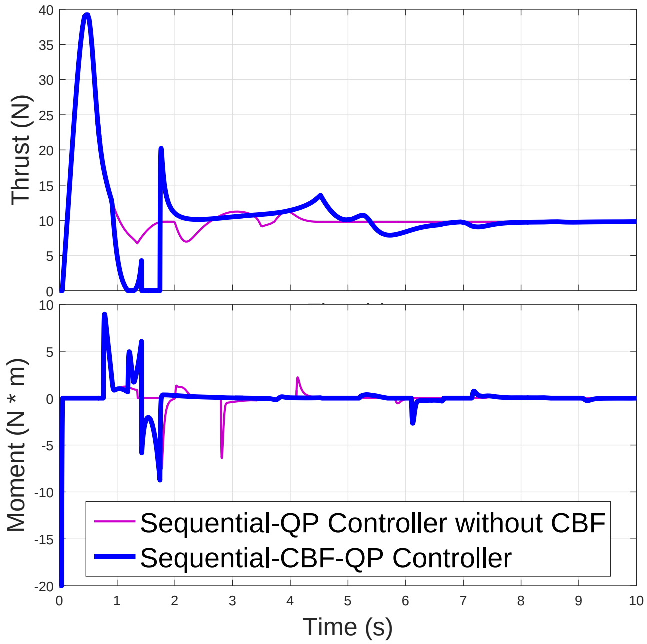

Fig. 4: Setpoint regulation with a single static obstacle (Fig.3): Plots of the thrust and moment with respect to time are illustrated for the Sequential-QP controller without CBF as well as the Sequential-CBF-QP controller. Both controllers are subject to a thrust and moment magnitude saturation. As can be seen, the resulting computed input for both controllers is only piece-wise continuous.

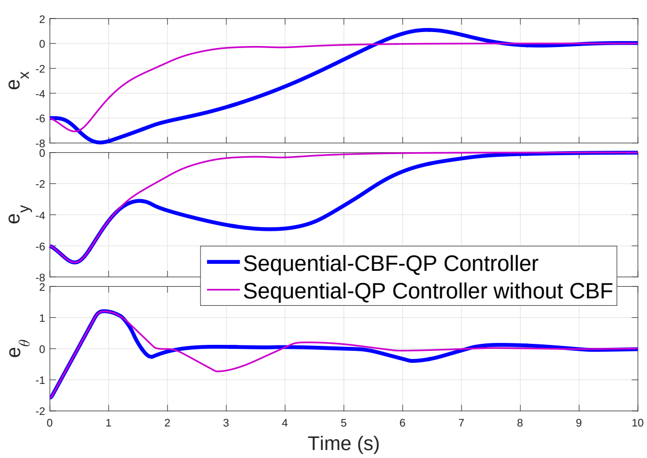

Fig. 5: Setpoint regulation with a single static obstacle (Fig.3): Translational and rotational error plots with respect to time are illustrated for the Sequential-QP controller without CBF and the Sequential-CBF-QP controller. As can be seen, the errors for the Sequential-CBF-QP controller increase midway through the plot. This is because the controller relaxes tracking to strictly enforce the safety constraint to prevent collision with the obstacle.

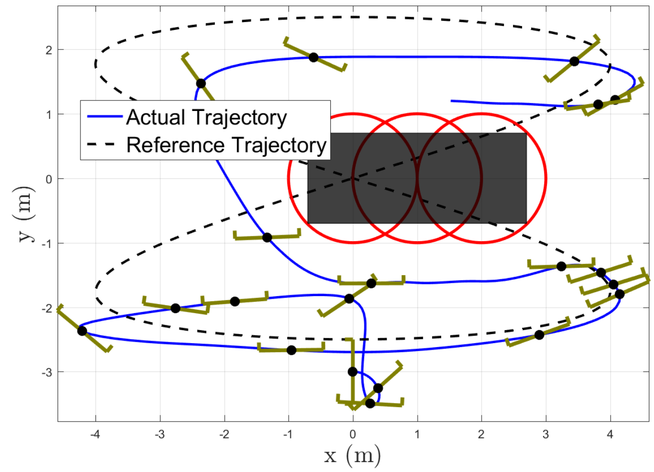

Fig. 6: Trajectory tracking with multiple static obstacles: Quadrotor position trajectory along with snapshots illustrating it's orientation are shown. This motion was obtained by the Sequential-CBF-QP controller when commanded to track a dynamic reference trajectory subject to multiple static obstacles. As can be seen, when the reference trajectory is no longer safe, the controller relaxes tracking and strictly enforces safety.

The resulting motion of the quadrotor is shown in Fig. 6. The quadrotor starts with an initial error and quickly tracks the time-varying reference trajectory. This occurs as long as the reference trajectory is in the safe region. Once the reference trajectory gets closer to the obstacles, the controller strictly enforces the safety guarantees and relaxes trajectory tracking automatically. In the figure, we see the quadrotor circumvents the obstacles and then converges towards the desired reference, before the reference trajectory goes through obstacles again. Snapshots of the quadrotor are also depicted along it's trajectory to illustrate the orientation of the quadrotor.

C. *Trajectory Tracking with a Single Dynamic Obstacle*

In this scenario, the Sequential-CBF-QP controller is commanded to track a dynamic reference trajectory while strictly enforcing safety guarantees for a time-varying obstacle. The reference trajectory is given as $(x_d, y_d) = [2.5\cos t, 2.5\sin t]$, the initial configuration of the quadrotor is given as $(x_0, y_0, \theta_0) = [0, 2, -\pi/2]$ and the initial velocity is zero.

Here, both the reference trajectory and the obstacle are time-varying. The chosen reference trajectory is a circle moving counterclockwise. The obstacle is also moving counterclockwise along this circle, albeit at a slower rate. Fig. 7 depicts the resulting motion of the quadrotor obtained due to the controller. Time is conveyed through color. The actual trajectory of the quadrotor as well as the obstacle positions at $t = 0$ are shown in grey, and as time progresses, the quadrotor trajectory becomes increasingly blue while the obstacle snapshots become increasingly red. As can be seen, the controller tracks the reference trajectory when safe and

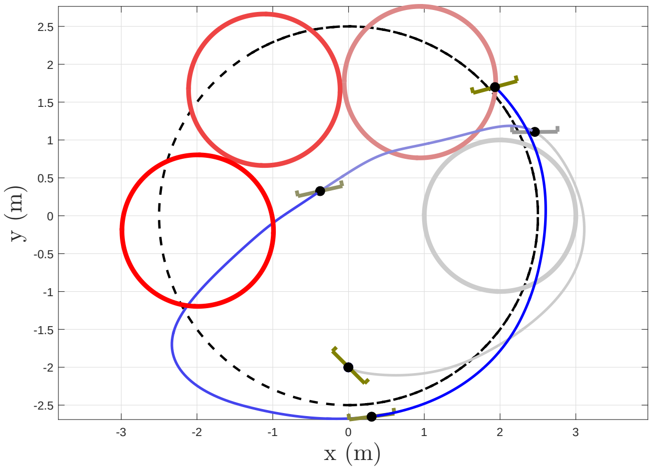

Fig. 7: Trajectory Tracking with a Single Dynamic Obstacle: Quadrotor position trajectory along with snapshots illustrating both the quadrotor's orientation as well as the position of the moving obstacle are shown. Time is conveyed in the figure through color. Both the quadrotor trajectory and the obstacle snapshots start in a grey color at $t = 0$ and become increasingly blue and red respectively. The controller is able to track the dynamic reference trajectory (black dotted circle) while subject to a time-varying safety constraint.

guarantees invariance of the time-varying safe set.

VII. Conclusion

We have presented a constrained controller design for a planar quadrotor to track a desired trajectory while simultaneously keeping the system state within a safety region. The controller is a sequential quadratic program, wherein the first level QP computed the desired thrust as a virtual input for position and the second level QP compute the quadrotor's thrust and moment physical inputs using the virtual input. An augmented control barrier function was introduced as means to guarantee safety in the position space while explicitly accounting for the quadrotor's orientation. Numerical results validating the proposed controller are demonstrated to achieve (a) setpoint regulation subject to a single static obstacle, (b) trajectory tracking with multiple static obstacles, and (c) trajectory tracking with a dynamic obstacle.

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