Exponential Control Barrier Functions for Enforcing High Relative-Degree Safety-Critical Constraints

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***Abstract*— We introduce Exponential Control Barrier Functions as means to enforce strict state-dependent high relative degree safety constraints for nonlinear systems. We also develop a systematic design method that enables creating the Exponential CBFs for nonlinear systems making use of tools from linear control theory. The proposed control design is numerically validated on a relative degree 6 linear system (the serial cart-spring system) and on a relative degree 4 nonlinear system (the two-link pendulum with elastic actuators.)**

I. Introduction

Safety-critical constraints are dynamical constraints that require the forward-invariance of a safe set, defined as super-level sets of scalar constraint functions. Control design to enforce these strict constraints for nonlinear systems is challenging. Enforcing constraints with high relative degree is harder.

A few control techniques to handle input and state-based constraints involve Model Predictive Control (MPC) [9], LMI optimizations [6], reference governors [3], [8], optimal control [10], [5], reachability analysis [10], [5], Barrier functions [17], [15], [16], [19], and other approaches [18], [6], [13]. However these methods typically involve one or more of the following shortcomings: not applicable to nonlinear systems, not applicable for real-time control, incapable of handling constraints with higher relative degree.

More recently, Control Barrier Functions (CBFs) [1] were introduced to convert safety constraints into a state-dependent linear inequality constraints on the inputs for application to adaptive cruise control. This method has been extended to general Riemannian manifolds in [20]. This approach combines control Lyapunov functions (CLFs) for stability [2] and CBFs for safety and solves a state-dependent quadratic program (QP) pointwise in time for the control input [4]. Combined control Lyapunov-Barrier functions have also been created by uniting CLFs and CBFs [14].

To address safety constraints with higher relative degree, the method of control Barrier functions was extended to position-based constraints with relative degree 2 in [20], [11]. Furthermore, a backstepping based method to design CBFs with higher relative degree was also introduced in [7]. However, achieving a backstepping based CBF design for higher relative degree systems (greater than 2) is challenging and has not been practically demonstrated. Here we present an alternate design method to address high relative degree constraints. Our method offers a simpler design process, compared to [7], for creating the control barrier functions for arbitrary high relative degree constraints. Moreover, as we will see, our method is a generalization of the preliminary approach in [20].

In this paper, we present a novel method called "Exponential Control Barrier Functions" (ECBFs) that can handle state-dependent constraints effectively for nonlinear systems with any relative degree. The design is based on the properties of linear control theory and therefore conventional methods such as pole placement control can be used to design ECBF constraints. The main contributions of the paper with respect to prior work are as follows:

- Introduction of Exponential Control Barrier Functions for enforcing high relative degree safety constraints.
- Formal construction of the Exponential CBFs based on techniques from linear control theory.
- Numerical validation of the Exponential CBFs to enforce safety constraints on a nonlinear system with relative degree 4 (the two-link pendulum with elastic actuators), and on a linear system with relative degree 6 (the serial cart-spring system).

The rest of the paper is organized as follows. Section II revists Control Barrier Function and Control Lyapunov Function based Quadratic Programs (CBF-CLF-QPs). Section III introduces the proposed method of Exponential Control Barrier Functions. Section IV presents numerical validation on different applications. Finally, Section V provides concluding remarks.

II. Control Lyapunov Function and Control Barrier Function based Quadratic Programs Revisited

A. Model

Consider the nonlinear control affine model

$$\dot{x} = f(x) + g(x)u, \quad (1)$$

$$y = h(x), \quad (2)$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^m$ is the control output.

Let r be the relative degree of control output y, we have:

$$y^{(r)} = L_f^r h(x) + L_g L_f^{r-1} h(x) u. \quad (3)$$

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The system is Input-Output linearizable if the decoupling matrix $L_g L_f^{r-1} h(x)$ is invertible, and we can apply an Input-Output linearizing controller:

$$u(x, \mu) = u^* + (L_g L_f^{r-1} h(x))^{-1} \mu, \tag{4}$$

where,

$$u^*(x) = -(L_g L_f^{r-1} h(x))^{-1} L_f^r h(x). \tag{5}$$

Defining the transverse variable

$$\eta = \begin{bmatrix} h(x) \\ L_f h(x) \\ L_f^2 h(x) \\ \dots \\ L_f^{r-1} h(x) \end{bmatrix}, \tag{6}$$

the linearized system becomes:

$$\dot{\eta} = \bar{f} + \bar{g}\mu, \quad \text{with} \quad \bar{f} = F\eta, \ \bar{g} = G, \tag{7}$$

where,

$$F = \begin{bmatrix} O & I & . & . & . & O \\ O & O & I & . & . & O \\ . & . & . & & & \\ O & . & . & . & . & I \\ O & . & . & . & . & O \end{bmatrix} \text{ and } G = \begin{bmatrix} O \\ . \\ . \\ . \\ I \end{bmatrix}. \tag{8}$$

B. Control Lyapunov Function based Quadratic Programs

A control approach based on control Lyapunov functions, introduced in [2], provides guarantees of exponential stability for the traverse variables η. In particular, a function $V(\eta)$ is an *exponentially stabilizing control Lyapunov function (ES-CLF)* for the system (1) if there exist positive constants $c_1, c_2, \lambda > 0$ such that

$$c_1 \|\eta\|^2 \le V(\eta) \le c_2 \|\eta\|^2, \tag{9}$$

$$\dot{V}(\eta, \mu) + \lambda V(\eta) \le 0. \tag{10}$$

Choosing a quadratic CLF candidate, $V(\eta) = \eta^T P \eta$, its time derivative can be computed as

$$\dot{V}(\eta, \mu) = L_{\bar{f}} V(\eta) + L_{\bar{g}} V(\eta)\mu, \tag{11}$$

where,

$$L_{\bar{f}} V(\eta) = \eta^T (F^T P + PF)\eta,$$
$$L_{\bar{g}} V(\eta) = 2\eta^T P G. \tag{12}$$

We can formulate the CLF condition in (10) into a quadratic program (QP) to solve for the control input pointwise in time. This also enables the incorporation of additional constraints. The CLF-based quadratic program (CLF-QP) [4] is as follows:

CLF-QP:

$$\mu^* = \underset{\mu, d}{\text{argmin}} \quad \mu^T \mu + p d^2 \tag{13}$$

$$\text{s.t.} \quad \psi_0(\eta) + \psi_1(\eta)\ \mu \le d \qquad \textbf{(CLF)}$$

$$A_c(x)\mu \le b_c(x) \qquad \textbf{(Constraints)}$$

where,

$$\psi_0(\eta) := L_{\bar{f}} V(\eta) + \lambda V(\eta),$$
$$\psi_1(\eta) := L_{\bar{g}} V(\eta). \tag{14}$$

C. Control Barrier Function

Consider an affine control system (1) with the goal to design a controller to keep the state x in the set

$$\mathcal{C} = \{x \in \mathbb{R}^n \mid h(x) \ge 0\}, \tag{15}$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function. Then a function $B : \mathcal{C} \to \mathbb{R}$ is a Control Barrier Function (CBF) [1] if there exists class $\mathcal{K}$ functions α_1 and α_2 such that, for all $x \in Int(\mathcal{C}) = \{x \in \mathbb{R}^n : h(x) > 0\}$,

$$\frac{1}{\alpha_1(h(x))} \le B(x) \le \frac{1}{\alpha_1(h(x))}, \tag{16}$$

$$\dot{B}(x, u) = L_f B(x) + L_g B(x) u \le \frac{\gamma}{B(x)}. \tag{17}$$

From [1], the important properties of the CBF condition in (17) is that if there exists a Control Barrier Function, $B : \mathcal{C} \to \mathbb{R}$, then $\mathcal{C}$ is forward invariant, or in other words, if $x(0) = x_0 \in \mathcal{C}$, i.e., $h(x_0) \ge 0$, then $x = x(t) \in \mathcal{C}, \forall t$, i.e., $h(x(t)) \ge 0, \forall t$. We can then incorporate the CBF condition into the CLF-QP controller as follow [1]:

CBF-CLF-QP:

$$\mu^* = \underset{\mu, d}{\text{argmin}} \quad \mu^T \mu + p d^2 \tag{18}$$

$$\text{s.t.} \quad \psi_{0,\varepsilon}(\eta) + \psi_{1,\varepsilon}(\eta)\ \mu \le d \qquad \textbf{(CLF)}$$

$$\psi_0^b(x) + \psi_1^b(x)\ \mu \le 0 \qquad \textbf{(CBF)}$$

$$A_c(x)\mu \le b_c(x) \qquad \textbf{(Constraints)}$$

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where

$$\psi_0^b(x) := L_f B(x) + L_g B u^*(x) - \frac{\gamma}{B(x)}, \tag{19}$$

$$\psi_0^b(x) := L_g B(x)(L_g L_f^{r-1} h(x))^{-1},$$

with $u^*(x)$ as defined in (5).

III. Exponential Control Barrier Function

Having revisited control Lyapunov and control Barrier function based quadratic programs, in this section, we introduce "Exponential Control Barrier Functions" (ECBFs) and present a novel method to systematically design a exponential CBF for high relative degree constraints. The term Exponential CBF is used since the resulting CBF constraint is an exponential function of the initial condition. Furthermore, the design and enforcement of ECBFs is based on linear control theory, and as a result, we can easily take advantage of conventional linear control design techniques such as pole placement.

Definition 1: (*Exponential Control Barrier Function*): Consider the dynamical system (1) and the set $\mathcal{C}_0 = \{x \in \mathbb{R}^n \mid B(x) \ge 0\}$, where $B : \mathbb{R}^n \to \mathbb{R}$ has relative degree

r_b. $B(x)$ is an exponential control Barrier function (ECBF) if there exists $K_b \in \mathbb{R}^{r_b \times 1}$ s.t.,

$$\inf_{u \in U} [L_f^{r_b} B(x) + L_g L_f^{r_b - 1} B(x) u + K_b \eta_b(x)] \geq 0, \forall x \in \mathcal{C}_0, \tag{20}$$

and $B(x(t)) \geq C_b e^{A_b t} \eta_b(x_0) \geq 0$, when $B(x_0) \geq 0$, where, the matrix A_b is dependent on the choice of K_b, and

$$\eta_b(x) = \begin{bmatrix} B(x) \\ \dot{B}(x) \\ \ddot{B}(x) \\ \vdots \\ B^{(r_b-1)}(x) \end{bmatrix} = \begin{bmatrix} B(x) \\ L_f B(x) \\ L_f^2 B(x) \\ \vdots \\ L_f^{r_b-1} B(x) \end{bmatrix}, \tag{21}$$

$$C_b = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{22}$$

Remark 1: Note that K_b, and consequently A_b, need to satisfy certain properties. We will develop these properties later in this section.

For the simple case of relative degree 1 ECBFs, the above definition can be reformulated as the following:

Definition 2: (*Relative degree 1 Exponential Control Barrier Function*): Consider the dynamical system (1) and the set $\mathcal{C}_0 = \{x \in \mathbb{R}^n \mid B(x) \geq 0\}$, where $B : \mathbb{R}^n \to \mathbb{R}$ has relative degree 1 and is continuously differentiable. $B(x)$ is an exponential control Barrier function (ECBF) with relative degree 1 if there exists $k_b \in \mathbb{R}$ s.t.,

$$\inf_{u \in U} [L_f B(x) + L_g B(x) u + k_b B(x)] \geq 0, \forall x \in \mathcal{C}_0, \tag{23}$$

and $B(x(t)) \geq B(x_0) e^{-k_b t} \geq 0$, when $B(x_0) \geq 0$.

Remark 2: Note that with relative degree 1, K_b and A_b in Definition 1 become scalar k_b, a_b and the condition for $B(x)$ to be an Exponential CBF is $a_b = k_b > 0$.

Remark 3: (*Relation between Exponential CBF and Zeroing CBF*): The relative degree 1 ECBF is a Zeroing CBF (ZCBF), as defined in [21] since $k_b \circ B(x)$ is a class $\mathcal{K}$ function of $B(x)$, and thus retains all the robustness properties of the ZCBF, as detailed in [21, Sec. 2.2].

We will next introduce the notion of Virtual Input-Output Linearization followed by the design of an Exponential CBF.

A. Virtual Input-Output Linearization

As mentioned in Section II, CLFs are an effective tool to handle stability for both linear and nonlinear systems. Furthermore, there is a systematic way to design CLFs for regulating outputs with arbitrary relative degree $r \geq 1$. If we can derive the CBF to the same form as a CLF, by using another Input-Ouput linearization for the CBF (17), we can then develop a general CBF for constraints with arbitrary relative degree $r_b \geq 1$. However, input-output linearizing $\dot{B}(x)$ is not directly feasible due to: (a) the decoupling matrix ($L_g B(x)$ when $r_b = 1$) being a vector and obviously not invertible, and (b) the control input u in (4) being already used to Input-Output linearize the output dynamics resulting in (7).

In order to solve this problem, we introduce the notion of Virtual Input-Ouput Linearization (VIOL) where an invertible decoupling matrix is not required and the control input μ is chosen to satisfy both the CLF condition (10) as well as input-output linearize the Barrier dynamics as follows. For a CBF with relative degree 1, let's define a virtual control input μ_b as follows:

$$\dot{B}(x, \mu) = L_f B(x) + L_g B(x) u(x, \mu) =: \mu_b(x, \mu). \tag{24}$$

Note that μ_b is a scalar. The CBF condition (17) then simply becomes:

$$\mu_b(x, \mu) \leq \frac{\gamma}{B(x)}. \tag{25}$$

We can then let a QP compute μ, μ_b so as to simultaneously satisfy the CLF condition (10) as well as both (24) and (25):

CBF-CLF-QP:

$$\mu^* = \underset{\mu, \mu_b, d_1}{\text{argmin}} \quad \mu^T \mu + p_1 \ d_1^2 \tag{26}$$

$$\text{s.t.} \quad \dot{V}(\eta, \mu) + \lambda V(\eta) \leq d_1 \qquad \textbf{(CLF)}$$

$$\mu_b - \frac{\gamma}{B(x)} \leq 0 \qquad \textbf{(CBF)}$$

$$A_c(x) \mu \leq b_c(x) \qquad \textbf{(Constraints)}$$

$$\dot{B}(x, \mu) = \mu_b. \qquad \textbf{(VIOL)}$$

Remark 4: Note that the solutions of the two controllers (26) and (18) are identical. However, the VIOL in the above CBF-CLF-QP opens up an effective and systematic way of designing the exponential CBFs.

B. Designing Exponential Control Barrier Functions

Consider the closed set $\mathcal{C}_0 = \{x \in \mathbb{R}^n \mid B(x) \geq 0\}$, with our control goal being to find an input u that guarantees forward invariance of $\mathcal{C}_0$, i.e., if $x_0 := x(0) \in \mathcal{C}_0$ then find u s.t., $B(x(t)) \geq 0, \forall t \geq 0$.

1) Simple Case: We first consider the problem of $B(x)$ having relative degree 1. Using VIOL, we have,

$$\dot{B}(x, \mu) = \mu_b. \tag{27}$$

If we want to drive $B(x)$ to zero, we can simply apply

$$\mu_b = -k_b B(x), \ k_b > 0 \tag{28}$$
$$\implies B(x(t)) = B(x_0) e^{-k_b t} \to 0 \text{ as } t \to \infty.$$

Making use of this property, we can guarantee the inequality constraint $B(x) \geq 0$ by imposing:

$$\mu_b \geq -k_b B(x), \ k_b > 0 \tag{29}$$

$$\implies \dot{B}(x, \mu) \geq -k_b B(x) \tag{30}$$

$$\implies B(x(t)) \geq B(x_0) e^{-k_b t} \geq 0 \text{ when } B(x_0) \geq 0. \tag{31}$$

Now, we can develop this approach for the general case with $B(x)$ having relative degree $r_b \geq 1$.

2) General Case: Let $r_b \geq 1$ be the relative degree of $B(x)$. We have,

$$\begin{aligned} B^{(r_b)}(x,\mu) &= L_f^{r_b} B + L_g L_f^{r_b-1} B \ u(x,\mu) \\ &= L_f^{r_b} B + L_g L_f^{r_b-1} B \\ &\quad \Big(u^*(x) + (L_g L_f^{r-1})^{-1} h(x)\mu \Big), \end{aligned} \tag{32}$$

where we substitued for $u(x,\mu)$ from (4). Further, using VIOL we have

$$B^{(r_b)}(x,\mu) = \mu_b, \tag{33}$$

such that the Input-Output linearized system becomes

$$\begin{cases} \dot{\eta}_b(x) &= F_b \eta_b(x) + G_b \mu_b \\ B(x) &= C_b \eta_b(x), \end{cases} \tag{34}$$

where $\eta_b(x)$ is as defined in (21), $F_b \in \mathbb{R}^{r_b \times r_b}, G_b \in \mathbb{R}^{r_b \times 1}$ are as defined below

$$F_b = \begin{bmatrix} 0 & 1 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ . & . & . & & & \\ 0 & . & . & . & . & 1 \\ 0 & . & . & . & . & 0 \end{bmatrix}, \quad G_b = \begin{bmatrix} 0 \\ . \\ . \\ . \\ 1 \end{bmatrix}, \tag{35}$$

and C_b is as defined in (22).

From this controllable canonical form, if we want to drive $B(x)$ to zero, we can easily find a pole placement controller $\mu_b = -K_b \eta_b$ with all negative real poles $p_b = -\begin{bmatrix} p_1 & p_2 & \dots & p_{r_b} \end{bmatrix}$, with $p_i > 0, \ i = 1, \cdots, r_b$, that obtains the closed loop matrix $A_b = F_b - G_b K_b$ with all negative real eigenvalues.

Motivated by (28), we can then apply,

$$\mu_b \geq -K_b \eta_b, \tag{36}$$

$$\implies \dot{\eta}_b \geq A_b \eta_b. \tag{37}$$

Assuming $K_b = \begin{bmatrix} k_b^1 & k_b^2 & \cdots & k_b^{r_b-1} \end{bmatrix}$ and from the definition of η_b in (21), the last row of the above vector inequality (37) results in

$$B^{(r_b)}(x) \geq k_b^1 B(x) + k_b^2 \dot{B}(x) + \cdots + k_b^{r_b-1} B^{(r_b-1)}(x). \tag{38}$$

This inequality constraint can also be written in terms of the pole locations p_i. To do this, we first define a family of outputs $y_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \cdots, r_b$, as follows,

$$y_i(x) = (\frac{d}{dt} + p_1) \circ (\frac{d}{dt} + p_2) \circ \cdots \circ (\frac{d}{dt} + p_{r_b}) \circ B(x), \tag{39}$$

with $y_0(x) := B(x)$. Associated with these outputs, we define a family of closed sets for $i = 0, \cdots, r_b$, as

$$\mathcal{C}_i = \{x \in \mathbb{R}^n \mid y_i(x) \geq 0\}. \tag{40}$$

Remark 5: Note that from the definition in (39), the output functions can also be written recursively as,

$$y_i(x) = \dot{y}_{i-1}(x) + p_i \ y_{i-1}(x). \tag{41}$$

Remark 6: Note that because of choice of K_b resulting in poles p_b, (38) is identical to $y_{r_b}(x) \geq 0$.

Theorem 1: Suppose the closed-set $\mathcal{C}_{r_b}$ is forward invariant for the system (1), then the closed-set $\mathcal{C}_0$ is forward invariant whenever initially $x_0 \in \mathcal{C}_i$ for each $i = 0, \cdots, r_b$, and $p_i > 0$ for each $i = 1, \cdots, r_b$.

Before we prove Theorem 1, we note the following corollaries:

Corollary 1: Suppose $y_i(x_0) \geq 0$ and $p_i > 0$ for $i = 1, \cdots, r_b$, then $B(x(t)) \geq 0, \forall t > 0$ when $B(x_0) \geq 0$.

Proof: This follows directly from Theorem 1 and the definition of the family of outputs y_i in (39) and closed sets $\mathcal{C}_i$ in (40). ∎

Corollary 2: Suppose $p_i \geq \max(-\frac{\dot{y}_{i-1}(x_0)}{y_{i-1}(x_0)}, \delta), \delta > 0$, for $i = 1, \cdots, r_b$, then $B(x(t)) \geq 0, \forall t > 0$ when $B(x_0) \geq 0$.

Proof: This follows from Corollary 1 and the fact from (41) that

$$y_i(x_0) \geq 0 \Leftrightarrow p_i \geq -\frac{\dot{y}_{i-1}(x_0)}{y_{i-1}(x_0)}.$$

∎

Proposition 1: Suppose the closed-set $\mathcal{C}_i$ is forward invariant for the system (1), then the closed-set $\mathcal{C}_{i-1}$ is forward invariant whenever initially $x_0 \in \mathcal{C}_i$, $x_0 \in \mathcal{C}_{i-1}$ and $p_i > 0$.

Proof: The proof essentially follows from [20, Prop. 1]. Since $x_0 \in \mathcal{C}_i$, then by forward invariance of $\mathcal{C}_i$, we have $x(t) \in \mathcal{C}_i$, for $t \in [0, \infty)$. From the definition of $\mathcal{C}_i$ and (41), this is equivalent to

$$\dot{y}_{i-1}(x(t)) + p_i \ y_{i-1}(x(t)) \geq 0, \quad \forall t \in [0, \infty).$$

Since the trajectory starts in $\mathcal{C}_{i-1}$, consider the extreme case when the system trajectory reaches the boundary of $\mathcal{C}_{i-1}$ at time T. Then, $y_{i-1}(x(T)) = 0$. However, according to the previous inequality, it follows that

$$\dot{y}_{i-1}(x(T)) \geq 0,$$

which means that the trajectory would never escape $\mathcal{C}_{i-1}$. ∎

We are now ready to prove Theorem 1.

Proof: (of Theorem 1:) The result follows from recursive application of Proposition 1. In particular, suppose $\mathcal{C}_{r_b}$ is forward invariant and $x_0 \in \mathcal{C}_{r_b}$, $x_0 \in \mathcal{C}_{r_b-1}, p_{r_b} > 0$, then from Proposition 1, $\mathcal{C}_{r_b-1}$ is forward invariant. Further if $x_0 \in \mathcal{C}_{r_b-2}$ and $p_{r_b-1} > 0$, then from Proposition 1, $\mathcal{C}_{r_b-2}$ is forward invariant. We can continue applying Proposition 1 so on to show $\mathcal{C}_0$ is forward invariant. ∎

Theorem 2: (*Main Result*): Suppose K_b is chosen s.t. A_b as defined in (22) is Hurwitz and total negative, and moreover $-\lambda_i(A_b) \geq -\frac{\dot{y}_{i-1}(x_0)}{y_{i-1}(x_0)}$. Then $\mu_b \geq -K_b \eta_b(x)$ with $\eta_b(x)$ as in (21), guarantees $B(x)$ is a Exponential CBF.

Proof: The choice of μ_b establishes the invariance of C_{r_b}, and K_b being Hurwitz and total negative ensures that the eigenvalues of A_b are real and negative. The rest follows from Corollary 2 and Theorem 1. ∎

In summary, if $B(x_0) \geq 0$ and the designed poles are chosen sufficiently small so that the condition in Corollary 2 holds, we can guarantee the state-dependent constraint $B(x) \geq 0$ by applying the exponential CBF condition (36).

Remark 7: (*Relation between Exponential CBF and Modified CBF with position-based constraints [20]*): For safety

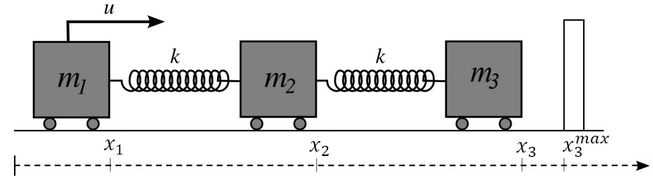


Fig. 1: The Serial Spring Mass System (Relative degree 6). The control goal is to drive the system state from the initial condition x_0 to a desired position for the 3^{rd} cart, x_{3d}, while strictly enforcing the safety constraint $x_3 \le x_3^{max}$. This system has three degrees-of-freedom and two degrees-of-underactuation.

constraints with relative degree 2, the modification of CBF presented in [20] extends the CBF condition for position-based (relative degree 2) constraints of the form $g(x) \ge 0$ by enforcing the standard CBF condition [1] on $h(x) = (\frac{d}{dt} + \gamma_b) \circ g(x) \ge 0, \gamma_b > 0$. Since, $h(x) \ge 0 \Leftrightarrow \dot{g}(x) \ge -\gamma_b g(x)$, enforcing the modified CBF $h(x) \ge 0$ results in $g(x) \ge 0$. $g(x)$ can then be seen as a relative degree 2 Exponential CBF.

We now present a QP that incorporates an Exponential CBF into a CLF-QP:

ECBF-CLF-QP:

$$\underset{\mu,\delta}{\text{argmin}} \quad \mu^T\mu + p\delta^2 \tag{42}$$

$$\begin{aligned} \text{s.t.} \quad & \dot{V}(\eta,\mu) + \lambda V(\eta) \le \delta && \textbf{(CLF)} \\ & A_C^{\mu}(x)\mu \le b_C^{\mu}(x) && \textbf{(Constraints)} \\ & \mu_b \ge -K_b\eta_b && \textbf{(Exponential CBF)} \\ & B^{(r_b)}(x,\mu) = \mu_b && \textbf{(VIOL)} \end{aligned}$$

In the next Section, we will validate our proposed method through two systems: serial spring mass system (linear system with relative degree 6) and a two-link pendulum with elastic actuators (nonlinear system with relative 4).

IV. Simulation Result

A. Serial Spring Mass System (Relative degree 6)

We validate our proposed method on a simple system comprising of serial masses connected through springs as shown in Fig. 1. The equations of motion for the system is as follows:

$$m_1\ddot{x}_1 = u + k(x_2 - x_1) \tag{43}$$

$$m_2\ddot{x}_2 = k(x_1 - x_2) + k(x_3 - x_2) \tag{44}$$

$$m_3\ddot{x}_3 = k(x_2 - x_3) \tag{45}$$

Defining the system state $x = [x_1 \ x_2 \ x_3 \ \dot{x}_1 \ \dot{x}_2 \ \dot{x}_3]^T$, we have the linear system: $\dot{x} = Ax + Bu$, where,

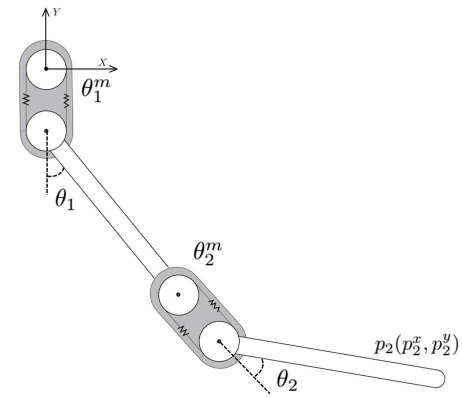


Fig. 2: The 2-link Pendulum with Elastic Actuators (Relative degree 4). The control goal is to drive the link angles from an initial configuration $\theta_1(0), \theta_2(0)$ to a desired configuration θ_{1d}, θ_{2d} while strictly enforcing the safety constraint on the vertical position of the end-effector, $p_2^y \ge p_{2min}$. This system is nonlinear with four degrees-of-freedom and two degrees-of-underactuation.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{k}{m_1} & \frac{k}{m_1} & 0 & 0 & 0 & 0 \\ \frac{k}{m_2} & -\frac{2k}{m_2} & \frac{k}{m_2} & 0 & 0 & 0 \\ 0 & \frac{k}{m_3} & -\frac{k}{m_3} & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} \tag{46}$$

Our control goal is to drive the system state from the initial condition x_0 to the desired position x_{3d} while considering the safety constraint $x_3 \le x_3^{max}$.

As illustrated in Fig.3, the CLF-QP controller violates the constraint with $max(x_3) = 3.27(m) > x_3^{max} = 3.15(m)$, the ECBF-CLF-QP controller with the desired poles $p_b = -0.12 \times \begin{bmatrix} 10 & 11 & 12 & 13 & 14 & 15 \end{bmatrix}$ can handle different safety constraints $x_3 \le x_3^{max}$.

B. 2-Link Pendulum with Elastic Actuators (Relative degree 4)

We consider a two-link pendulum with elastic actuators, as shown in Fig.2. The two-link pendulum is a nonlinear system. With elastic actuators, the commanded torques τ_1, τ_2 of the two motors will generate torques at two joints indirectly through the following motor dynamics:

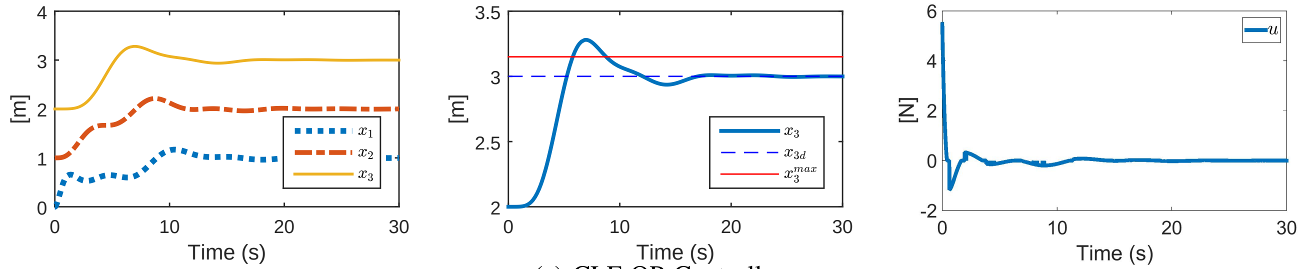
$$\begin{aligned} J_m\ddot{\theta}_1^m &= k(\theta_1 - \theta_1^m) + \tau_1, \\ J_m\ddot{\theta}_2^m &= k(\theta_2 - \theta_2^m) + \tau_2, \end{aligned} \tag{47}$$

where θ_1, θ_2 are joint angles of the robot, θ_1^m, θ_2^m are angles of two motors, J_m, k are inertia and stiffness of motors. Then, the torque at two joints of the robot would be:

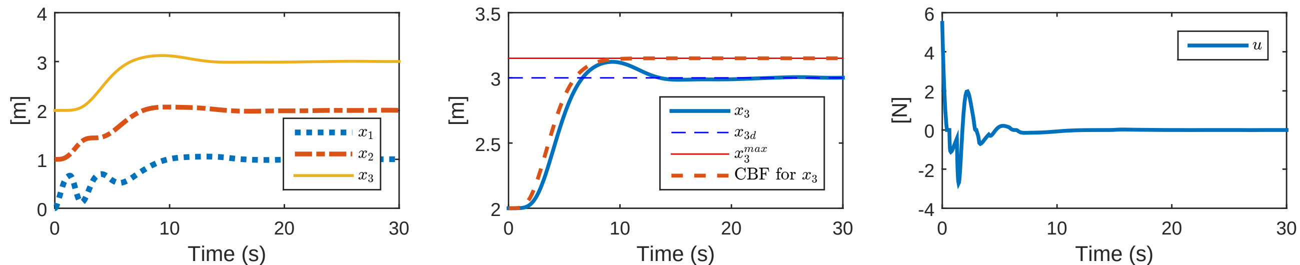
$$\begin{aligned} u_1 &= -k(\theta_1 - \theta_1^m) - \xi\dot{\theta}_1, \\ u_2 &= -k(\theta_2 - \theta_2^m) - \xi\dot{\theta}_2, \end{aligned} \tag{48}$$

where ξ is the damping coefficient at the joints.

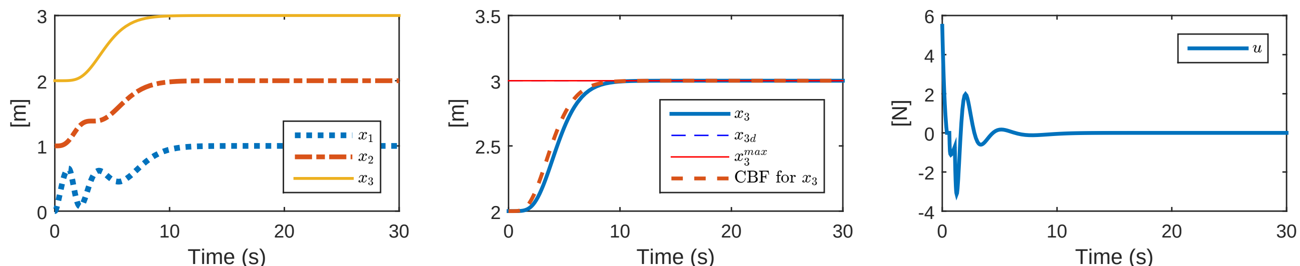
We apply the Exponential CBF-CLF-QP controller on the two-link pendulum nonlinear system with the above motor dynamics to enforce a constraint on the position of the end effector $p_2(p_2^x, p_2^y)$ (see Fig.4). While the nominal CLF-QP



(a) CLF-QP Controller



(b) $x_3^{max} - x_{3d} = 15(cm)$



(c) $x_3^{max} - x_{3d} = 0(cm)$

Fig. 3: Serial Spring Mass System: Cart positions and input force for the Exponential CBF-CLF-QP controller with desired cart position $x_{3d} = 3(m)$ and with safety constraint $x_3 \leq x_3^{max}$. The pole locations p_b for the CBF encode the performance specifications of enforcing a safety constraint. As seen above, varying the safety constraint while keeping the poles fixed keeps the peak forces and speed of system response the same. Simulation video: `http://youtu.be/okojFUtaiDk`.

controller violates the constraint, the ECBF-CLF-QP with poles $p_b = -\begin{bmatrix} 5 & 5.5 & 6 & 10 \end{bmatrix}$, can handle different safety constraints.

C. Discussion

Exponential CBFs have the advantage that they can be easily designed for high relative degree safety constraints using tools from linear control theory. The pole locations for the designed CBF and the poles used to design the CLF can be chosen to more smoothly tradeoff stability of tracking and enforcement of safety. Despite these advantages, Exponential Control Barrier Function have some limitations. The choice of pole location depends on initial conditions as stated in Corollary 2, requiring careful choice of these poles. Although, if the initial conditions are bounded, the poles can be chosen based on these bounds. Furthermore, the presented Exponential CBF-based control design is dependent on the system model and could be sensitive to model uncertainty. Preliminary results to address safety constraints with model uncertainty are presented in [12].

V. Conclusion

We have introduced Exponential Control Barrier Functions (ECBFs) as means to enforce high relative degree safety constraints for nonlinear systems. We have presented a systematic design method that enables creating the Exponential CBFs based on pole placement. The designed exponential CBFs along with control Lyapunov functions (CLFs) were formulated as a unified quadratic program. The proposed control design has been numerically validated on a relative degree 6 linear system (the serial cart-spring system) and on a relative degree 4 nonlinear system (the two-link pendulum with elastic actuators.)

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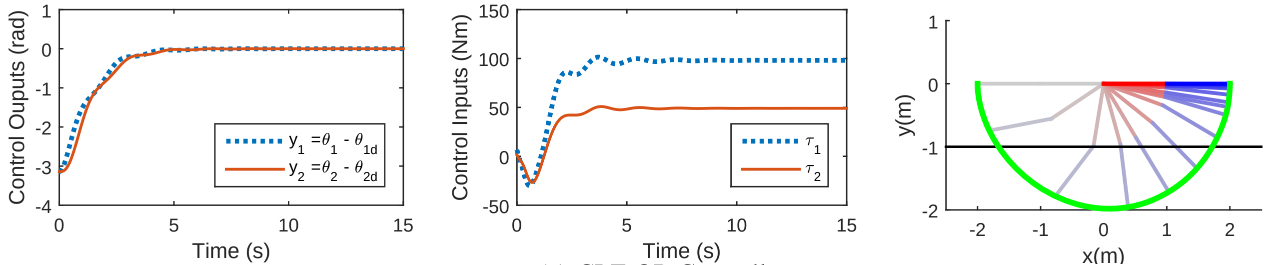
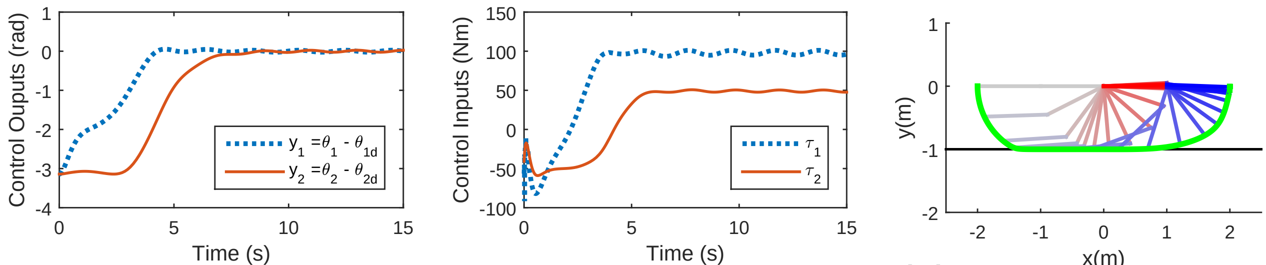
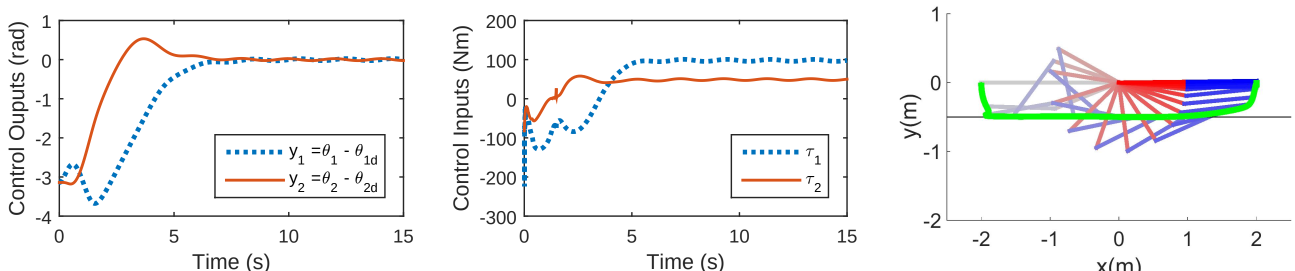


(a) CLF-QP Controller



(b) CBF-CLF-QP Controller with $p_{2min} = -1.0[m]$.



(c) CBF-CLF-QP Controller with $p_{2min} = -0.5[m]$.

Fig. 4: 2-link Pendulum with Elastic Actuators: The control goal is to drive the link angles from an initial configuration $\theta_1(0) = -\pi, \theta_2(0) = 0$ to a desired configuration $\theta_{1d} = \pi, \theta_{2d} = 0$ while strictly enforcing the safety constraint on the vertical position of the end-effector, $p_2^y \geq p_{2min}$. Note that the links have unit length and the controller has to aggressively move the links to enforce the strict safety constraint. Simulation video: `http://youtu.be/okojFUtaiDk`.

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