Safety-Critical and Constrained Geometric Control Synthesis using Control Lyapunov and Control Barrier Functions for Systems Evolving on Manifolds

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***Abstract*— Geometric controllers for mechanical system have been well-developed in literature. Although geometric controllers possess the advantages of being coordinate-free and compact, these controllers don't take into account constraints such as safety-critical collision constraints or input constraints. In this paper, we extend control Lyapunov and control Barrier function based quadratic programs to incorporate various constraints. We combine Control Lyapunov Function(CLF) and Control Barrier Function(CBF) through a relaxation. Qualitative analysis of this design method is derived in detail and we provide simulation results on Cartesian space, the spherical pendulum, and 3D pendulum.**

I. Introduction

Every real-world control problem has inherent constraints that needs to be satisfied. Constraints such as actuator constraints are prevalent. Moreover, safety-critical constraints are omnipresent, for instance we have several aerospace applications with critical distance constraints in the form of collision constraints, and orientation constraints in the form of inclusion and exclusion regions for pointing sensitive imaging and communication payloads. Developing controllers that are capable of enforcing these constraints, while providing formal guarantees on safety, as well as stabilizing non-equilibrium dynamic motions for systems evolving on complex manifolds, and being implementable on embedded processors with hard real-time constraints is a challenge.

A variety of approaches exist for incorporating constraints, for instance an overview of methods are presented for Cartesian systems in [17], these include model predictive control (MPC)-based [10], [13] which is typically hard to implement in real-time for nonlinear geometric controllers. Barrier functions have been used to obtain certificates with invariance of sets while tracking [6], [7], [14]–[16], [19], [20] which ensure invariance of level sets, but don't allow for the state to travel across the level set boundary, even when the safety region allows it. Several real-world satellite re-targeting with safety constraints have been considered, for instance [6], [7], [18] uses cone inclusion and exclusion constraints, [9], [16] employs logarithmic barrier potentials for control in the presence of input and output's saturations, and [19] adds barrier functions to the objective function to handle constrained MPC problems.

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Our strategy is to extend the recent work on control Lyapunov function based quadratic programs [5], and recent work on control barrier functions based quadratic programs [2] for adaptive cruise control. We will extend these methods from Euclidean space to a general manifold. This involves solving for a control feedback as an online optimization problem that enforces constraints on the Lyapunov candidate derivative [1], [12], as well as additional constraints representing input limits, safety constraints in the form of barrier function derivatives constraints, etc. In this way, by imposing the constraints on the control input, the control design is converted to the search of a cost function and solving a state-dependent optimization problem. However, the trade-off is that analysis of the system trajectory is intractable because of the embedded optimization scheme.

The main contribution of this paper is to extend the concept of control Lyapunov function and control Barrier function based quadratic programs to the geometric setting. This will serve to create a unified geometric controller framework that enforces constraints, offers guarantees on safety and stability, and be implementable with real-time constraints. The rest of the paper is organized as follows. Section II introduces fundamental properties of geometric control, CLFs, and CBFs, Section III develops a general form of CBFs for geometric mechnaical systems, and Section IV applies to simple mechnaical systems on $\mathbb{R}^3, \mathbb{S}^2$ and $SO(3)$ to better illustrate the effectiveness of the control design and performance.

II. Mathematical Preliminary

In this section, we are going to list some basic concepts about geometric control, control Lyapunov function and control barrier function. These concepts are necessary for the control design in later sections. We refer to [1], [2], [4] for a detailed discussion.

A. Elements of Geometric Control

This subsection introduces relevant basics of geometric control. Given a mechanical system which evolves on a sufficiently smooth manifold M, we denote its configuration variable as q, the tangent space at q as T_qM and the tangent bundle as $TM = \cup T_qM$. The state space representation of this system is then given by $(q, \dot{q}) \in TM$.

Further, a vector field is a mapping from each point $q \in M$ to a vector in the corresponding T_qM. While, an one form $\omega : T_qM \to \mathbb{R}$ defines a mapping from the tangent space at each point $q \in M$ to the set of real number. *The differential* of a smooth function $f : M \to \mathbb{R}$ is given as an one-form by the formula below:

$$\langle df, X\rangle_q = \lim_{t\to 0} \frac{f(\alpha(t)) - f(q)}{t}$$

where the curve $\alpha : [-1,1] \to M$ satisfies $\alpha(0) = q, \alpha'(0) = X$.

Similarly, if we have a smooth function $g : M \times M \to \mathbb{R}$, then denote the differential with respect to the i^{th} argument as $d_i g$ $(i = 1, 2)$. The corresponding definition is the same as the single-variable case *while keeping the other argument fixed.*

Now we will describe the properties of the specific system under study in this paper. This type of system is called simple mechanical systems. And we assume the reader already know about Riemannian geometry. If a system's configuration manifold M has the following structures:

1) The total inertia is given by a metric $M_q : T_qM \to T_q^*M$ which represents the kinetic energy by $\langle\langle \dot{q}, \dot{q}\rangle\rangle = \langle M_q\dot{q}, \dot{q}\rangle$.
2) A connection ∇ which is compatible with M_q which means that
$$\nabla_X \langle\langle Y, Z\rangle\rangle = \langle \nabla_X(M_qY), Z\rangle + \langle M_qY, \nabla_X Z\rangle$$
where X, Y, Z are vector fields. Here we assume that M is a Riemannian manifold and thus *the covariant derivative of one form or vector field is well-defined.*
3) A collection of one forms $F_j : T_qM \to \mathbb{R}$ representing a basis of external force applied where $j \in [1, m]$.
4) A smooth function $\mathcal{V}_q : M \to \mathbb{R}$ representing the potential energy.
5) A configuration error $\Psi : M \times M \to [0, \infty]$ that serves as a measure of distance between the two points $q, q_d \in M$ in the manifold M. We also require $\Psi(q, q_d)$ to be quadratic as defined in [4]. Then the differential $d_1\Psi$ could be used as the position error denoted as e_q.
6) A transport map $\mathscr{T}_{(q,q_d)} : T_{q_d}M \to T_qM$ which maps a tangent vector at q_d to one at q with the compatible condition,
$$d_2\Psi = -\mathscr{T}^*_{(q,q_d)} d_1\Psi,$$
where $\mathscr{T}^*_{(q,q_d)}$ is the dual map of $\mathscr{T}_{(q,d)}$. In this way, we are able to compare tangent vector in different tangent spaces as shown below:
$$e_{\dot{q}} = \dot{q} - \mathscr{T}_{(q,q_d)}\dot{q}_d.$$

Then the system is called a simple, fully actuated system with dynamics as:

$$\nabla_{\dot{q}}\dot{q} = M_q^{-1}(-d\mathcal{V}_q(q) + \sum_{j=1}^{m} F_j(q, \dot{q})u^j) \tag{1}$$

with $u^j \in \mathbb{R}$. Also, given a dynamically feasible reference $q_d(t) \in M$, a general expression of Lyapunov function could be given as:

$$V = \alpha\Psi(q, q_d) + \frac{1}{2}\langle\langle e_{\dot{q}}, e_{\dot{q}}\rangle\rangle + \varepsilon\langle e_q, e_{\dot{q}}\rangle.$$

B. Geometric Control Lyapunov Function

We start with a control affine system in $\mathbb{R}^n$ of the form,

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ x(t_0) &= x_0, \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

Then, for system (2), a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is called Exponentially Stabilizing Lyapunov Function (ES-CLF) if there exist constants $c_1, c_2, c_3 > 0$ such that

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$
$$\inf_{u\in\mathbb{R}^m} \{L_fV + L_gVu + c_3V\} \le 0$$

Next we extend the idea of a CLF to a system described by (1). Instead of Lie derivative, we use differential for its definition. So a smooth function $V : TM \to \mathbb{R}$ is called the geometric ES-CLF of the simple mechanical system if there exist constants c_1, c_2, c_3 such that

$$\begin{aligned} &c_1\langle\langle d\Psi_1, d\Psi_2\rangle\rangle \le V(q, \dot{q}) \le c_2\langle\langle d\Psi_1, d\Psi_2\rangle\rangle, \\ &\inf\nolimits_{u\in\mathbb{R}^m}\{\underbrace{\langle d_1V, \dot{q}\rangle - \langle d_1V, M_q^{-1}d\mathcal{V}_q\rangle}_{\text{equivalent to } L_fV} \\ &+ \underbrace{\textstyle\sum_{j=1}^m \langle d_1V, M_q^{-1}F_j\rangle}_{\text{equivalent to } L_gV} u^j + c_3V\} \le 0, \end{aligned} \tag{3}$$

where the control input $u = [u^1, u^2, \cdots, u^m]^T$. As we can see, CBF is

C. Geometric Control Barrier Function

Next, we establish a CBF for a geometric mechanic system. As introduced in [2], CBFs can only be defined with respect to a region in the state space TM. For the system (2), suppose we have a continuously differentiable $h : \mathbb{R}^n \to \mathbb{R}$ and the region is defined by the level set $\mathcal{C} = \{x \in \mathbb{R}^n : h(x) \ge 0\}$. Then a smooth function $B : \mathcal{C} \to \mathbb{R}$ is called a CBF of $\mathcal{C}$ if there exist two class $\mathcal{K}$ function α_1, α_2 and $\eta > 0$ so that

$$\frac{1}{\alpha_1(h(x))} \le B(x) \le \frac{1}{\alpha_2(h(x))},$$
$$\inf_{u\in\mathbb{R}^m} \{L_fB + L_gBu - \frac{\eta}{B}\} \le 0,$$

for any $x \in \mathcal{C}^\circ$ which is the interior of $\mathcal{C}$.

Similarly, for the system (1), there exists a smooth function $h : TM \to \mathbb{R}$ and the region $\mathcal{C} = \{(q, \dot{q}) \in TM : h(q, \dot{q}) \ge 0\}$ has nonempty interior. Then a smooth function $B : TM \to \mathbb{R}$ is defined to be a geometric CBF of $\mathcal{C}$ if there exist two class $\mathcal{K}$ functions and constant $\eta > 0$ so that

$$\begin{aligned} &\tfrac{1}{\alpha_1(h(q,\dot{q}))} \le B(q, \dot{q}) \le \tfrac{1}{\alpha_2(h(q,\dot{q}))}, \\ &\inf\nolimits_{u\in\mathbb{R}^m}\{\langle d_1B, \dot{q}\rangle - \langle d_1B, M_q^{-1}d\mathcal{V}_q\rangle \\ &\quad + \textstyle\sum_{j=1}^m \langle d_1V, M_q^{-1}F_i\rangle u^j - \frac{\eta}{B}\} \le 0, \end{aligned} \tag{4}$$

for any $x \in \mathcal{C}^\circ$.

III. Control Barrier Function for Geometric Mechanical Systems

In this section, we will propose a general process to generate new constraint functions based on given constraints on configuration variable q. Following this, a candidate CBF is constructed and combined with a candidate CLF in the geometric control setting. The combined controller will be used to realize tracking under these constraints.

A. Constrained Tracking Problem Formulation

Given the mechanical system (1) and a list of physical constraints on the configuration variables as defined below, for $i \in \{1, 2, \cdots, l\}$

$$g_i = (-1)^{\delta_i}(b_i - \Psi(q, q_i)), \quad \delta_i \in \{0, 1\}, \tag{5}$$

$$\mathcal{B}_i = \{(q, \dot{q}) \in TM : \ g_i(q) \geq 0, \quad q_i \in M\}, \tag{6}$$

where the set $\mathcal{B}_i$ is the *feasible region* for constraint g_i. Then, $\mathcal{B} = \cap_{i=1}^{l} \mathcal{B}_i$, needs to satisfy the compatible condition,

$$\mathcal{B}^\circ \neq \varnothing$$

where $\mathcal{B}^\circ$ denotes the interior of the set $\mathcal{B}$ in M's topology.

***Remark** 1:* The constraints presented here are in terms of the configuration error. Moreover, we assume q_i is a constant.

***Remark** 2:* The compatible condition above ensures that the set of all specified constraints are feasible. Also, we use the value of δ_i to indicate whether the configuration q should stay close or far away from the center q_i, or equivalently whether the center q_i is safe or not.

Given a smooth reference curve $q_d(t) \in M$ for $t \in [0, \infty]$ with reference input $u_d \in \mathbb{R}^m$, the control goal is to design feedback input $u = u(t, q, \dot{q})$ so that the following conditions are satisfied:

$$q(t) \in \mathcal{B}, \ \forall t \in [0, \infty] \qquad \text{(Safety)}$$

$$q(t) \to q_d(t) \text{ when } (q_d, \dot{q}_d) \in \mathcal{B} \qquad \text{(Asymptotic Stability)}$$

B. Candidate CBF

Next, based on the given constraints in terms of only configuration variables, we expand the feasible region to the state space, i.e, the tangent bundle TM, as follows. Choose a smooth class $\mathcal{K}$ function $\alpha : [0, \infty) \to \mathbb{R}$, define the new state-dependent constraint functions as:

$$h_i(q, \dot{q}) = \gamma_i \alpha(g_i(q)) + \langle dg_i, \dot{q} \rangle,$$

which is well-defined since dg_i is an one-form on M and thus it's a *linear functional* of the tangent space at each point q.

If we define each feasible region as $\mathcal{C}_i = \{(q, \dot{q}) \in TM : h_i(q, \dot{q}) \geq 0\}$, then the new expanded region could be given as:

$$\mathcal{C} = \cap_i^l \mathcal{C}_i = \cap_i^l \{(q, \dot{q}) \in TM : \gamma_i \alpha(g_i(q)) + \langle dg_i \geq 0 \rangle\}$$

where we could tune the parameters γ_i to make its interior nonempty.

We next look at the relationship between this expanded region and the original feasible region. Intuitively, it should follow that $(q, \dot{q}) \in \mathcal{C} \Rightarrow q \in \mathcal{B}$. However, that's not the case since $g(q)$ could be negative and the variable q definitely doesn't lie in $\mathcal{C}$. But if we take the dynamics into account, the construction of h_i is meaningful according to the following proposition.

***Proposition** 1:* (Forward Invariance Preservation of the Feasible Region $\mathcal{B}$)

Suppose the region $\mathcal{C}$ is forward invariant for the system (1), then the region $\mathcal{B}$ is also forward invariant whenever initially $q_0 \in \mathcal{B}$ and $(q_0, \dot{q}_0) \in \mathcal{C}$.

Proof: Since $(q_0, \dot{q}_0) \in \mathcal{C}$, by forward invariance, it holds that $(q(t), \dot{q}(t)) \in \mathcal{C}$ which is equivalent to

$$\gamma_i \alpha(g_i(q(t))) + \langle dg_i, \dot{q}(t) \rangle|_{q(t)} \geq 0$$

for any $i \in \{1, 2, \cdots, l\}$ and $t \in [0, \infty)$

Consider the extreme case when the system trajectory reaches the boundary of $\mathcal{B}$ as $\partial\mathcal{B}$ at t_1, then $g(q(t_1)) = 0$. According to the previous inequality, it follows that

$$\frac{dg_i}{dt}(q(t_1)) = \langle dg_i, \dot{q}(t_1) \rangle|_{q(t_1)} \geq 0,$$

which means that the trajectory would never escape $\mathcal{B}_i$.

Therefore, by definition of $\mathcal{B}$, any system trajectory would never escape the region $\mathcal{B}$, i.e, it's forward invariant. ■

***Remark** 3:* This proposition guarantees that if we could render the region $\mathcal{C}$ forward invariant instead, the physical constraints are satisfied automatically. Thus, we only need to consider the situation when $g_i(q) \geq 0$ and $h_i(q, \dot{q}) = \gamma_i \alpha(g_i(q)) + \langle dg_i, \dot{q} \rangle$.

Having generated a satisfying region $\mathcal{C}$, we propose the following CBF candidate for each constraint in terms of h_i:

$$B_i(q, \dot{q}) = \frac{1}{h_i(q)}, \quad (q, \dot{q}) \in \mathcal{C}^\circ.$$

Differentiating each B_i with respect to time yields:

$$\begin{aligned} \dot{B}_i(q, \dot{q}) = & -\frac{1}{h_i^2}\dot{h}_i(q, \dot{q}) \\ = & -\frac{1}{h^2}(\gamma_i \alpha' \langle dg_i, \dot{q} \rangle + \langle \nabla_{\dot{q}} dg_i, \dot{q} \rangle) \\ & + \frac{1}{h^2}\langle dg_i, M_q^{-1} d\mathcal{V}_q \rangle - \frac{1}{h^2}\sum_{j=1}^{m} \langle dg_i, M_q^{-1} F_j \rangle u^j \end{aligned}$$

where M_q denote the metric tensor of inertia.

In order to make each B_i become a CBF, the following condition must be satisfied:

$$\inf_{u \in \mathbb{R}^m} \{\dot{B}_i - \frac{\mu^i}{B_i}\} \leq 0, \quad \forall (q, \dot{q}) \in \mathcal{C}$$

which is equivalent to that

$$\inf_{u \in \mathbb{R}^m} \{-\dot{h}_i - \mu^i h_i^3\} \leq 0, \qquad \forall (q, \dot{q}) \in \mathcal{C}$$

where $\mu^i > 0$.

To present this concisely, we introduce the following notation for each $i \in \{1, 2, \cdots, l\}$:

$$\phi_0^i(q, \dot{q}) := [\langle dg_i, M_q^{-1} F_1 \rangle, \cdots, \langle dg_i, M_q^{-1} F_m \rangle]^T,$$

and $\phi_1^i(q, \dot{q})$ as the sum of all terms in $\dot{h}_i$ that don't contain

u. Then, the condition which the CBF must satisfy can be written in a concise form:

$$-(\phi_0^i \cdot u + \phi_1^i + \mu_i h_i^3) \le 0, \quad i \in \{1, 2, \cdots, l\}$$

Note that this condition can only fail when $\phi_0^i = 0$ and $\phi_1^i + \mu_i h_i^3 < 0$. Using the fact that system (1) is *fully-actuated* and that metric is *non-degenerate*, the condition of $\phi_0^i = 0$ is equivalent to $dg_i = (-1)^{\delta_i+1} e_q(q, q_i) = 0$ in the cotangent space. We assume that the set $\mathcal{D}_i = \{q \in \mathcal{B} : e_q(q, q_i) = 0\}$ has *measure zero* in M, then by sub-additivity of measure, it follows that

$$\mathcal{D}_{CBF} = \cup_{i=1}^m \mathcal{D}_i \Rightarrow m(\mathcal{D}_{CBF}) \le \sum_{i=1}^m m(\mathcal{D}_i) = 0$$

Hence, the CBFs defined here hold everywhere except for a set with measure zero. So they can be treated almost globally valid.

C. Candidate ES-CLF

From Section II, a candidate CLF has the form

$$V(t, q, \dot{q}) = \alpha \Psi(q, q_d) + \frac{1}{2} \langle\langle e_{\dot{q}}, e_{\dot{q}} \rangle\rangle + \varepsilon \langle e_q, e_{\dot{q}} \rangle$$

where the coefficients $\alpha, \varepsilon > 0$ are specifically chosen to make this form quadratic.

Differentiating it with respect to t gives us an expression which include the control input explicitly as:

$$\begin{aligned}\dot{V} =& \alpha \langle d_1 \Psi, e_{\dot{q}} \rangle - \langle\langle e_{\dot{q}}, [\frac{d}{dt}\Big|_{q \text{ fixed}} (\mathscr{T} \dot{q}_d) + (\nabla_{\dot{q}} \mathscr{T}) \dot{q}_d] \rangle\rangle \\ &+ \varepsilon [\langle \nabla_{e_{\dot{q}}}(e_q), e_{\dot{q}} \rangle - \langle d_1 \Psi, ((\nabla_{e_{\dot{q}}} \mathscr{T}) e_{\dot{q}}) \rangle] \\ &- \varepsilon \langle e_q, [\frac{d}{dt}\Big|_{q \text{ fixed}} (\mathscr{T} \dot{q}_d) + (\nabla_{\dot{q}} \mathscr{T}) \dot{q}_d] \rangle \\ &- [\varepsilon \langle d_1 \Psi, M_q^{-1} d\mathcal{V}_q \rangle + \langle\langle e_{\dot{q}}, M_q^{-1} d\mathcal{V}_q \rangle\rangle] \\ &+ \sum_{j=1}^m [\varepsilon \langle d_1 \Psi, M_q^{-1} F_j u^j \rangle + \langle\langle e_{\dot{q}}, M_q^{-1} F_j u^j \rangle\rangle]\end{aligned}$$

For V to be a CLF, we require the following condition to be satisfied:

$$\inf_{u \in \mathbb{R}^m} \{\dot{V} + \eta V\} \le 0.$$

This can be written concisely by defining

$$\psi_0(q, \dot{q}) = \begin{bmatrix} [\varepsilon \langle e_q, M_q^{-1} F_1 \rangle + \langle\langle e_{\dot{q}}, M_q^{-1} F_1 \rangle\rangle] \\ [\varepsilon \langle e_q, M_q^{-1} F_2 \rangle + \langle\langle e_{\dot{q}}, M_q^{-1} F_2 \rangle\rangle] \\ \vdots \\ [\varepsilon \langle e_q, M_q^{-1} F_n \rangle + \langle\langle e_{\dot{q}}, M_q^{-1} F_n \rangle\rangle], \end{bmatrix}$$

and $\psi_1(q, \dot{q})$ as the net term in $\dot{V} + \eta V$ that doesn't depend on u. The above condition can then be written concisely as:

$$\psi_0 \cdot u + \psi_1 + \eta V \le 0,$$

where $\eta > 0$ is the lower bound of convergent rate. Note that this condition can only fail when $\psi_0 = 0$ and $\psi_1 > 0$. We also assume that the region $\mathcal{D}_{CLF} = \{(q, \dot{q}) \in \mathcal{C} : \psi_0(q, \dot{q}) = 0\}$ has measure zero. Hence almost global property also holds for the CLF. Thus, the condition of CLF and CBF would only fail to be satisfied for the set $\mathcal{D} = \mathcal{D}_{CLF} \cup \mathcal{D}_{CBF}$ and is again of measure zero.

D. Optimization-based Controller Design

The previous subsections introduce CBF and CLF for the general mechanical system which hold almost globally. Now we are able to put them all together into an optimization scheme.

First, decompose the total control input into two part, feed-forward and feedback parts:

$$u = u_{ff} + u_{fb}$$

where the feed-forward term is directly computed as the solution of the linear equation below

$$\sum u_{ff}^j F_j = d\mathcal{V}_q(q) + M_q [\frac{d}{dt}\Big|_{q \text{ fixed}} (\mathscr{T} \dot{q}_d) + (\nabla_{\dot{q}} \mathscr{T}) \dot{q}_d]$$

which comes from geometric control theory.

Then compute the feedback term u_{fb} based on the following state-dependent optimization problem.

(CLF-CBF-QP Control Design)

Minimize the cost function

$$J = \frac{1}{2} v^T H v + \frac{1}{2} \lambda \delta^2$$

with optimization parameters $[v^T, \delta]$ subject to

$$\psi_0 \cdot v + [\psi_0 \cdot u_{ff} + \psi_1 + \eta V] \le \delta \quad \text{(Relaxed Stability)}$$

$$-(\phi_0^i \cdot v + \psi_0^i \cdot u_{ff} + \phi_1^i + \mu_i h_i^3) \le 0 \quad \text{(Strict Safety)}$$

where the weighting matrix H is positive definite and $\lambda > 0$.

Then assign part of the solution to the feedback term u_{fb}:

$$u_{fb} = v \in \mathbb{R}^m$$

For the case when there's no feasible solution, set $u_{fb} = 0$.

Remark 4: As discussed in [2], the constraints imposed by CBF are treated as *Hard Constraints* that must be satisfied for all the time while the tracking has to be compatible with this safety concerns. That's why we put it in a relaxed form.

Remark 5: In order to guarantee the existence and uniqueness of system trajectory, we require the control input should be at least Lipschitz continuous in x. We refer to [12] for a detailed discussion on the solution's continuity of quadratic programming.

The proposition below shows some property of this controller.

Proposition 2: (Safety Critical Property of the State-dependent QP Controller)

If the following conditions are satisfied:

- The initial condition $(q_0, \dot{q}_0)$ stay within the region's interior $\mathcal{C}^\circ$.
- The singularity set $\mathcal{D}$ where CBF or CLF fail has measure zero and $\mathcal{D} \in \mathcal{C}^\circ$.
- The embedded quadratic programming has feasible solution for the set $\mathcal{C} \backslash \mathcal{D}$.

then the safety-critical goal is satisfied by the controller in subsection (III-D).

Proof: If a solution exists for quadratic programming , by property of convex programming this solution is unique. So by assumption this control input is well-defined and Lipschitz continuous for the set $\mathcal{C}$.

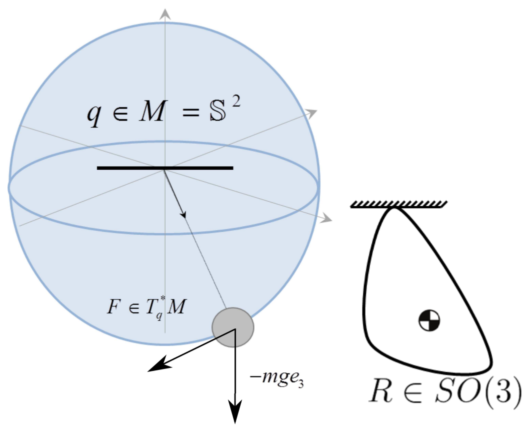


Fig. 1: Diagram of the spherical and 3D pendulums.

Since the set $\mathcal{D}$ has measure zero, it has an empty interior. Hence, the system's trajectory can only traverse it at discrete time points. For the time period (t_1, t_2) when $(q, \dot{q}) \in \mathcal{C} \backslash \mathcal{D}$, all the hard constraints are satisfied by the controller. Applying Theorem 2 in [2] yields that system trajectory would never escape $\mathcal{C} \backslash \mathcal{D}$. When $(q, \dot{q}) \in \mathcal{D}$, a solution might not exist but the current state lies in C° since $\mathcal{D} \in C^{\circ}$. For both cases, the system state always stay within $\mathcal{C}$. And thus it's forward-invariant for the system (1).

Using Proposition (1), it follows that the feasible region $\mathcal{B}$ is also forward-invariant. ∎

***Remark** 6:* Though this proposition require lots of conditions, none of them is quite strong. The first one comes naturally for physical systems. The second one requires the set where singularity exists has measure zero. As shown in [11], this type of almost global property is common for geometric control. Thus, it could be treated as an intrinsic property of the specific manifold. The last one could be satisfied by tuning the parameter γ_i in CBF's definition. Since the singularity set only has zero measure, it won't cause any trouble for numerical simulation.

IV. Application to Some Simple Mechanical Systems

In this section, we present several concrete examples to better illustrate this CLF-CBF-QP design and validate its effectiveness in making a good trade-off between convergent rate and safety. The systems under study are a point mass, a spherical pendulum and a 3D pendulum which have been studied in [3], [4], [8].

A. Safety-Critical Control Design of Point Mass

This system has a single free point mass driven by the external force. Its configuration space is $\mathbb{R}^3$. Select its Cartesian coordinate as the configuration variable $q = (x, y, z)^T$. The tangent space at each point is also $\mathbb{R}^3$ and the system dynamics is written as:

$$\ddot{q} = m^{-1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

with metric $\langle\langle \dot{q}_1, \dot{q}_2 \rangle\rangle = m \dot{q}_1 \cdot \dot{q}_2$.

The configuration error is simply given in terms of Euclidean norm

$$\Psi(q, q_d) = \frac{1}{2} \|q - q_d\|^2 = \frac{1}{2}(q - q_d) \cdot (q - q_d)$$

equipped with the differential:

$$e_q = d_1 \Psi(q, q_d) = q - q_d, \quad d_2 \Psi(q, q_d) = -(q - q_d)$$

With this type of configuration error, each obstacle represents the sphere or its complement. The compatible transport map is given by the identity, i.e,

$$\mathscr{T}_{(q, q_d)} \dot{q}_d = \dot{q}_d, \quad e_{\dot{q}} = \dot{q} - \dot{q}_d$$

since we could directly compare two tangent vector in flat spaces like $\mathbb{R}^3$.

Given a desired trajectory q_d, the feed-forward force is computed as $u_{ff} = m \ddot{q}_d$. The CBF and CLF candidates are written as:

$$\begin{aligned} B_i &= (-1)^{\delta_i} \{\gamma_i (b_i - \frac{1}{2}(q - q_i)^2) - (q - q_i) \cdot \dot{q}\} \\ V &= \frac{1}{2} m (\dot{q} - \dot{q}_d) \cdot (\dot{q} - \dot{q}_d) + \frac{1}{2} \alpha \|q - q_i\|^2 \\ &\quad + \varepsilon (q - q_i) \cdot (\dot{q} - \dot{q}_i) \end{aligned}$$

where we take the function $\alpha(x) = x$.

Also, it is obvious that the singular set is given by:

$$\mathcal{D} = \{(q, \dot{q}) : q = q_i, \dot{q} = \dot{q}_d + q_i - q_d, \; i \in [1, l], t \in [0, \infty)\}$$

which is the union of several curves in TM and thus has measure zero.

Based on the previous discussion, we now are able to perform the numerical simulation. In this simulation, we compare several controllers which are the original geometric control, the geometric CLF-QP and the geometric CBF-CLF-QP. In the simulation environment, the point mass is confined to move in the space between two spheres. And we consider an extreme case when a reference trajectory mistakenly traverse the inner sphere as the obstacle. The results in Fig 2 show that the CBF-CLF-QP controller would execute a very large force to pull the point mass away from boundary which is reasonable according to property of barrier function.

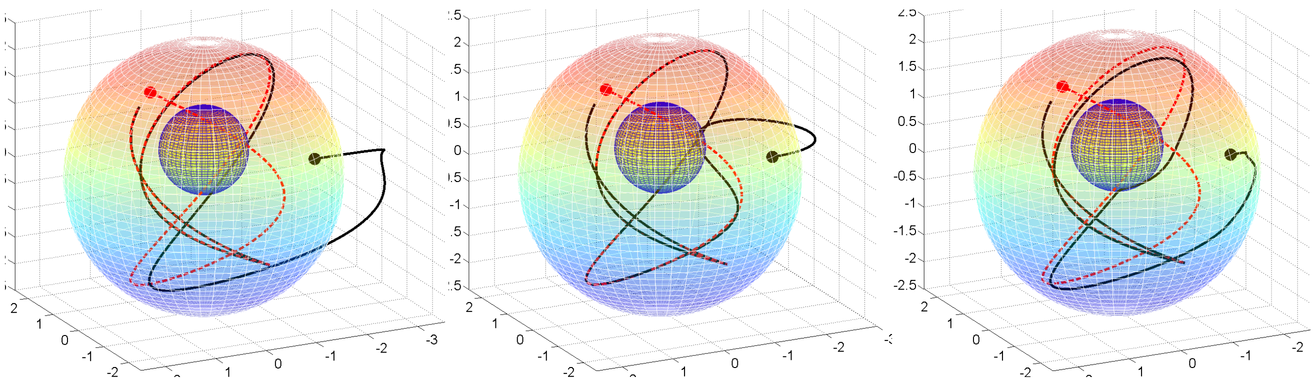
B. Safety-Critical Control Design of Spherical Pendulum

The system under consideration here is a spherical pendulum comprised of a point mass connected to a pivot through a rigid rod. The configuration of this system is given by $\mathbb{S}^2$ as shown in Fig 1. Using the directional vector q of this point mass' displacement, we have the dynamics equation as:

$$\begin{aligned} \dot{q} &= \omega \times q \\ \dot{\omega} &= q \times (\tfrac{F}{ml} - \tfrac{g}{l} e_3) \end{aligned} \quad \text{or} \quad \ddot{q} + (\dot{q} \cdot \dot{q}) q = -\hat{q}^2 (\frac{F}{ml} - \frac{g}{l} e_3)$$

which is a fully-actuated simple system since the tangential force could span the tangent space at each point.

Since we nondimensionalize the system already, the metric degenerates to the inner product $\langle\langle \dot{q}_1, \dot{q}_2 \rangle\rangle = \dot{q}_1 \cdot \dot{q}_2$. The



(a) Geometric Min-Norm Control (b) Original Geometric Control (c) Geometric CLF-CBF-QP Control

Fig. 2: Simulation of various controllers on the point mass system on $\mathbb{R}^3$, which is required to track a desired trajectory while being restricted to move within the region between two spheres. As can be seen for (a) min-norm, and (b) geometric control, the system trajectory exits the outer sphere as well as enters the inner sphere, violating critical safety region constraints. However, for (c) CLF-CBF-QP controller, the critical safety constraint is enforced while still following the desired trajectory.

configuration error utilized here is defined as:

$$\Psi(q, q_d) = 1 - q \cdot q_d, \quad e_q = d_1\Psi = \hat{q}^2 q_d, \quad d_2\Psi = \hat{q}_d^2 q$$

where the hat map $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ converts a three-dimensional vector to a skew-symmetric real matrix.

The compatible transport map and velocity error are given by:

$$\mathscr{T}_{(q,q_d)}\dot{q}_d = (q_d \times \dot{q}_d) \times q, \quad e_{\dot{q}} = \dot{q} - (q_d \times \dot{q}_d) \times q$$

Thus, the corresponding CLF and CBF are given as

$$B_i = (-1)^{\delta_i}\{\gamma_i(q \cdot q_i - b_i) + q_i \cdot \dot{q}\}$$
$$V = \frac{1}{2}e_{\dot{q}} \cdot e_{\dot{q}} + \frac{1}{2}\alpha(1 - q \cdot q_d) + \varepsilon e_q \cdot e_{\dot{q}}$$

In order to analyze the singularity set $\mathcal{D}$, we write out the vector explicitly

$$\varphi_0^i = (-1)^{\delta_i} q_i$$
$$\psi_0^i = (e_{\dot{q}} + \varepsilon e_q)$$

The singularity set of spherical pendulum has similar structure as that of the point mass. It is defined as:

$$\mathcal{D} = \{(q, \dot{q}) \in TM : q = q_i, e_{\dot{q}} + \varepsilon e_q = 0\}$$

The simulation results of this spherical pendulum are shown in Fig 3. In this process, the spherical pendulum is required to stay within the gap between two cones. As shown in this figure, three types of controllers including the geometric CLF min-norm, the original geometric control and CLF-CBF-QP on $\mathbb{S}^2$ are also compared. The first two haven't considered any safety and thus the pendulum would escape the feasible region. While the CLF-CBF-QP controller helps realize a safety-critical tracking on the configuration manifold.

C. Safety-Critical Control Design of 3D Pendulum

The mechanical system considered here is a rigid body attached to a pivot. Its configuration space is given by $SO(3)$ which is a matrix Lie group. Since the Lie algebra $\mathfrak{so}(3)$ is uniquely identified with $\mathbb{R}^3$, we could still treat the tangent space as $\mathbb{R}^3$. The system dynamics is given by the Euler equation:

$$\dot{R} = R\Omega$$
$$\dot{\Omega} = J^{-1}(J\Omega \times \Omega + M)$$

where $R \in \mathbb{R}^{3\times3}$ and $\Omega \in \mathbb{R}^3$.

Thus this system is fully actuated since the number of independent forces equal to the manifold's dimension. The metric on the tangent space is given by $\langle\langle\Omega_1, \Omega_2\rangle\rangle = \Omega_1^T J \Omega_2$ where $\Omega_1, \Omega_2 \in \mathbb{R}^3$. The configuration error is given by:

$$\Psi(R, R_d) = \frac{1}{2}tr(I - R_d^T R), \quad e_R = \frac{1}{2}(R_d^T R - R^T R_d)^\vee$$

where the map $\cdot^\vee : \mathfrak{so}(3) \to \mathbb{R}^3$ is the inverse of $\hat{\cdot}$.

Then the compatible velocity error is computed as:

$$e_\Omega = \Omega - R^T R_d \Omega_d$$

where the desired body angular velocity $\Omega_d = R_d^T \dot{R}_d$.

The CBF and CLF for this case are shown below:

$$B_i = (-1)^{\delta_i}\{\gamma_i[\frac{1}{2}tr(R_i^T R) - b_i] + \tilde{S}_i \cdot \Omega\}$$
$$V = \frac{1}{2}e_\Omega \cdot J e_\Omega + \frac{1}{2}\alpha \cdot tr(I - R_d^T R) + \varepsilon e_R \cdot e_\Omega$$

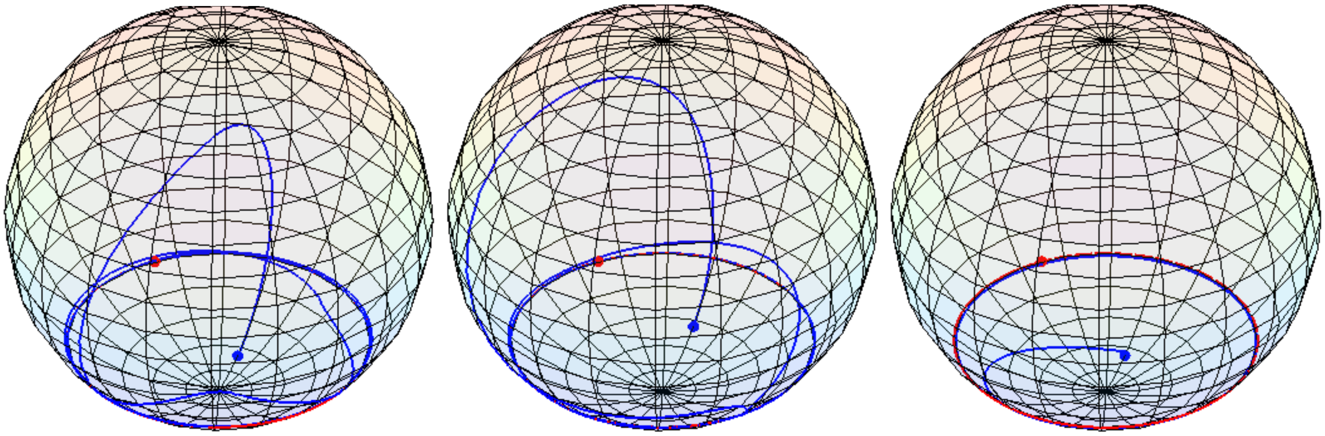
where the vector $\tilde{S}$ is defined as below:

$$\tilde{S} = \begin{bmatrix} \tilde{R}_{23}^i - \tilde{R}_{32}^i \\ \tilde{R}_{31}^i - \tilde{R}_{13}^i \\ \tilde{R}_{12}^i - \tilde{R}_{21}^i \end{bmatrix}, \text{ where } \tilde{R}^i = R_i^T R$$

Since the visualization of $SO(3)$ is not quite intuitive, we haven't included any figure about simulation results.

V. Conclusion

The major contribution of this paper lies in extending the notion of CLF and CBF to a general manifold of simple, fully-actuated mechanical systems and developing a



(a) Geometric CLF Min-Norm Control (b) Original Geometric Control (c) Geometric CLF-CBF-QP Control

Fig. 3: Simulation of various controllers on the spherical pendulum system on S^2, restricted to remain in a cone inclusion area. As can be seen, for (a) geometric CLF min-norm, and (b) original geometric control, the system trajectory makes a wide excursion, going outside the cone area defined by the circular reference trajectory, whereas for (c) CLF-CBF-QP, the controller ensures the trajectory remains within a cone region while converging to the desired trajectory.

coordinate-free and safety-critical controller based on them. By specifying the feasible region in terms of configuration error, we are able to expand it to a larger one in the state space where CBF could be directly applied. Then using the quadratic Lyapunov function in geometric control as a CLF, a state-dependent quadratic programming is set up with constraints imposed by CLF and CBF. With the given solution as the feedback term, the controller is able to provide critical guarantee on safety while also maintaining asymptotic stability. This is demonstrated in the simulations of point mass and spherical pendulums systems.

Although we have considered only fully actuated systems here, the proposed methodology can also be applied to underactuated systems. We can also incorporate constraints that are more complex, than the simple input constraints considered here.

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